Volume hyperbolicity and wildness

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**Abstract**

It is known that *volume hyperbolicity* (partial hyperbolicity and uniform expansion or contraction of the volume in the extremal bundles) is a necessary condition for robust transitivity or robust chain recurrence hence for tameness. In this paper, on any 3-manifold we build examples of quasi-attractors which are volume hyperbolic and wild at the same time. As a main corollary, we see that, for any closed 3-manifold $M$, the space $\mathrm{Diff}^1(M)$ admits a non-empty open set where every $C^1$-generic diffeomorphism has no attractors or repellers.

The main tool of our construction is the notion of flexible periodic points introduced in [BS]. For ejecting the flexible points from the quasi-attractor, we control the topology of the quasi-attractor using the notion of filtrating Markov partition, which we introduce in this paper[1].

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# 1 Introduction

## 1.1 Backgrounds

Hyperbolic dynamical systems are those which are nowadays considered as well understood (see [BDV] for further information). However, they are far to represent the majority of dynamical systems: it is known from the late sixties that there are open sets in $\mathrm{Diff}^r(M)$ of non-uniformly hyperbolic systems. These systems are each unstable. Hence it seems improbable that we obtain a complete topological classification of them.

Nevertheless, some recent works propose a panorama of non-hyperbolic dynamical systems (see [B, CP]). They attempt to characterize the robust non-hyperbolicity by local phenomena (for instance robust heterodimensional cycle or/and robust homoclinic tangencies) and weak hyperbolic structures (such as partial hyperbolicity or dominated splittings).

As a first step of this global program, there is an important division of the space of dynamical systems into two classes of completely different nature:

- We say that a diffeomorphism is *tame* if each chain recurrence class is $C^1$-robustly isolated. This implies that there are finitely many chain recurrence classes and that the number of classes does not vary under perturbations of the system.

- We say that a diffeomorphism is *wild* if it admits a neighborhood in $\mathrm{Diff}^1(M)$ in which $C^1$-generic systems have infinitely many chain recurrence classes.

The result in [BC] shows that the set of tame diffeomorphisms and the set of wild diffeomorphisms are disjoint $C^1$-open sets whose union is dense in $\mathrm{Diff}^1(M)$ (see [A] for a preliminary version, which only holds for $C^1$-generic diffeomorphism).

While the uniform hyperbolicity is a sufficient condition for tameness, there are examples of non-hyperbolic tame diffeomorphisms [S, M, BD1, BV]. We also have examples of wild systems [BD2, BD3, BLY], which are necessarily non-hyperbolic. However, this division is far to be understood:

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- Every chain recurrence class (which is neither a sink nor a source) of tame systems has a structure called *volume hyperbolicity*: it consists of a finest dominated splitting $E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_k$ whose extremal bundles $E_1$ and $E_k$ are volume contracting and expanding, respectively. See [BDP, B].
- All the known examples of wild diffeomorphisms use lack of dominated splittings which allow us to produce sinks or sources (possibly after restriction of the dynamics on a normally hyperbolic invariant submanifold).

The aim of this paper is to approach the frontier between tame and wild dynamics, by producing examples of wild diffeomorphisms under a setting of volume hyperbolicity, using a completely different method from the existing ones.

We consider diffeomorphisms on 3-manifolds which have a chain recurrence class $C(p)$ (where $p$ is a hyperbolic saddle of $s$-index 2, i.e., $\dim(W^s(p)) = 2$) admitting a partially hyperbolic splitting $E^{cs} \oplus_< E^u$, where $E^u$ is uniformly expanding and $E^{cs}$ is uniformly area contracting (for the precise definitions of chain recurrence classes and partially hyperbolic splittings, see Section 2.3). We assume that $C(p)$ is not contained in a normally hyperbolic invariant surface (otherwise $p$ is an attracting point in restriction to this surface, and $C(p)$ is trivial). Note that, according to [BC2], this is equivalent to that there is a periodic point $q$ in $C(p)$ for which the strong unstable manifold of $q$ meets $C(p)$ at a point $q' \neq q$. This is the setting for the examples of robustly transitive diffeomorphisms in [BV] and of the robustly transitive attractor of [Ca]. In this setting, we will nevertheless build examples of wild diffeomorphisms.

Following the definition of wild homoclinic classes in [B], we say that a chain recurrence class $C(p)$ of a diffeomorphism $f$ is ($C^1$-)*wild* if it is locally generically non-isolated, that is, there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of $f$ and a residual set $\mathcal{R} \subset \mathcal{U}$ such that for every $g \in \mathcal{R}$ the continuation $C(p_g)$ is not isolated (every neighborhood of $C(p_g)$ has non-empty intersection with a chain recurrence class which is not equal to $C(p_g)$). Note that having a wild chain recurrence class is a sufficient condition for the wildness of a diffeomorphism.

We will construct a diffeomorphism having a wild chain recurrence class under the above setting with additional conditions. Note that the volume hyperbolicity implies that there are neither sinks nor sources in a sufficiently small neighborhood of $C(p)$. For some of our examples, the class $C(p)$ will be the unique quasi-attractor in an attracting neighborhood, and its basin will cover a residual subset of this attracting neighborhood.

The main technique of our construction is the notion of *$\varepsilon$-flexible points* defined in [BS]. It is an $s$-index 2 periodic point $x$ admitting an $\varepsilon$-perturbation (with respect to a fixed $C^1$-distance) which changes $x$ into an $s$-index 1 periodic orbit whose stable manifold being an arbitrarily chosen curve in the center-stable plane. In [BS], we also showed that the existence of flexible points with arbitrarily small $\varepsilon$ is a prevalent phenomenon under the presence of a robustly non-hyperbolic 2-dimensional center-stable bundle with no dominated splitting. Therefore, these flexible points seem to be the good candidate for being ejected from the original chain recurrence class $C(p)$. We only need to choose a curve out of the class $C(p)$ and give a perturbation which turns the stable manifold of the periodic point into the chosen one. Once we have the disjointness of a fundamental domain of the stable manifold from $C(p)$, then we can see that the orbit of $x$ is indeed out of $C(p)$.

This simple argument transforms the question of knowing if $C(p)$ is tame or wild into an almost purely topological problem. The question is: *in the local center-stable manifold passing through $x$, does there exist a path starting from $x$ which is disjoint from $C(p)$?* In the examples of tame dynamics in [BV, Ca], the class is either the whole manifold or a Sierpinski carpet in the center-stable direction. These examples have flexible points. Meanwhile, the flexibility is not enough to ejecting the point, because there is no path going out of the class from the point.

Here we give a setting in which the intersection of the chain recurrence class with the center stable manifold passing through the flexible point will be contained in a finite union of disjoint discs. This condition allows us to choose the stable manifold of this flexible point so that it avoids these discs. In such a situation, the flexibility of the periodic point enables us to eject it from the original class.

Now, we state our result in more formal way.

### 1.2 Statement of the result

We consider a diffeomorphism $f$ of a 3-manifold. We will define precisely in the next section the notions of *partially hyperbolic filtrating Markov partition* of *saddle type* (Definition 2.5) or of *attracting type* (Definition 2.6). Let us give a rough description of it.

It consists of the following:

- a Markov partition $\{R_i\}_{i=1,\dots,k}$ whose *rectangles* $R_i$ are cylinders (we say that a subset of a 3-manifold is a *cylinder* if it is $C^1$-diffeomorphic to $\mathbb{D}^2 \times [0,1]$, where $\mathbb{D}^2$ is the unit disc in $\mathbb{R}^2$);
- the union of the rectangles $\mathbf{R} = \bigcup_{i=1}^{k} R_i$ is a *filtrating set*, that is, an intersection of an attracting region $A$ and a repelling region $R$ (in the attracting type, the repelling region $R$ is the whole manifold, in other words, the union of the rectangles is an attracting region);
- the *vertical boundary* of each rectangle (i.e., the side of the cylinder $(\partial\mathbb{D}^2) \times [0,1] = \mathbb{S}^1 \times [0,1]$) is contained in the boundary of the attracting region $\partial A$;
- for the saddle type filtrating Markov partition we require that the *horizontal boundaries* $\mathbb{D}^2 \times \partial[0,1]$ are contained in the boundary of the repelling region $\partial R$;
- there is a partially hyperbolic structure $E^{cs} \oplus_< E^u$ defined on the union of the rectangles: $Df$ (the differential map of $f$) leaves invariant a continuous *unstable cone field* $\mathcal{C}^u$ defined on the union of the rectangles, transverse to the horizontal discs $\mathbb{D}^2 \times \{t\}$ and containing the vertical lines $\{p\} \times [0,1]$.

We say that a hyperbolic periodic point $x \in R_i$ with *$s$-index* 2 has *large stable manifold* if there is a local stable manifold of $x$ which is a disc contained in $R_i$ (necessarily transverse to the unstable cone field) whose boundary is contained in the vertical boundary of $R_i$. Finally, recall that two hyperbolic periodic points $p$ and $q$ are said to be *homoclinically related* if $W^u(p)$ and $W^s(q)$ have non-empty transverse intersection and the same holds for $W^u(q)$ and $W^s(p)$.

We are now ready to state our main result.

**Theorem 1.** *Let $f$ be a diffeomorphism of a 3-manifold, having a partially hyperbolic filtrating Markov partition $\mathbf{R} = \bigcup_{i=1}^{k} R_i$ of either saddle type or attracting type. Assume that there is a periodic point $p \in \mathbf{R}$ and the following holds:*

- *$p$ is an $s$-index 2 hyperbolic periodic point with large stable manifold;*
- *there is an $s$-index 2 hyperbolic periodic point $p_1$ homoclinically related with $p$ having a complex (non-real) stable eigenvalue;*
- *there is a periodic point $q$ of $s$-index 1 which is $C^1$-robustly in the chain recurrence class $C(p,f)$ of $p$: for every $g$ which is sufficiently $C^1$-close to $f$ one has*
$$C(p_g, g) = C(q_g, g),$$
*where $p_g$ and $q_g$ are the continuation of $p$ and $q$ for $g$.*

*Then, there are a $C^1$-diffeomorphism $g$ arbitrarily $C^1$-close to $f$, hyperbolic periodic points $x_g$ and $y_g$ (of $g$) of $s$-index 1 and 2 respectively, such that the chain recurrence classes $C(x_g, g)$ and $C(y_g, g)$ are trivial, that is, they are equal to the orbits of $x_g$ and $y_g$ respectively. Furthermore, we can require that the Hausdorff distance between the orbit of $x_g$ (resp. $y_g$) and the chain recurrence class of $C(p,f)$ is arbitrarily small.*

In all the known examples of wild systems, as in [BD1], the process of creating a new class out of a given class consists of producing an attracting or a repelling region, for instance by changing the derivative of a periodic orbit, using Franks' Lemma. In our context, the partially hyperbolic structure prevents the existence of attracting regions in $\mathbf{R}$: The unstable manifold of every periodic point in $\mathbf{R} = \bigcup R_i$ cuts transversely the stable manifold of our starting point $p$ (which has large stable manifold). If furthermore the diffeomorphism is volume hyperbolic over $\mathbf{R}$ (i.e., contracts uniformly the area in $E^{cs}$), it prohibits the existence of repelling periodic orbits.

Indeed, such examples do exist: we also show that there are examples of chain recurrence classes which are volume hyperbolic and simultaneously satisfy the hypothesis of the Theorem 1:

**Proposition 1.1.** *Given a compact 3-manifold $M$, there are non-empty $C^1$-open sets $\mathcal{U}_{\rm sdl}$, $\mathcal{U}_{\rm att} \subset \mathrm{Diff}^1(M)$ such that every $f \in \mathcal{U}_{\rm sdl}$ (resp. $f \in \mathcal{U}_{\rm att}$) admits a partially hyperbolic filtrating Markov partition $\mathbf{R} = \bigcup_{i=1}^{k} R_i$ of saddle type (resp. of attracting type), periodic points $p, p_1$ and $q$ which satisfy the hypothesis of Theorem 1 such that the partially hyperbolic splitting $E^{cs} \oplus E^u$ over the maximal invariant set in $\mathbf{R}$ (i.e., $\cap_{n\in\mathbb{Z}} f^n(\mathbf{R})$) is volume hyperbolic.*

Theorem 1 announces the existence of an arbitrarily small perturbation of the initial diffeomorphism producing a saddle point with trivial chain recurrence class near the class of $p$. As we will see later, the hypothesis of Theorem 1 is $C^1$-open condition. The openness is clear except the persistence of partially hyperbolic filtrating Markov partitions, which will be discussed in Section 2. Thus, the production of new class can be done keeping the hypothesis. Then, by repeating it infinitely many times, we have the following:

**Corollary 1.2.** *Under the hypothesis of Theorem 1, there are a $C^1$-neighborhood $\mathcal{U}$ of $f$ and a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that, for every $g \in \mathcal{R}$ the class $C(p_g, g)$ is the Hausdorff limits of two sequences of periodic orbits $\mathcal{O}(x_i)$ and $\mathcal{O}(y_i)$ of $s$-index 2 and 1 respectively and having trivial classes, that is,*

$$C(x_i, g) = \mathcal{O}(x_i) \text{ and } C(y_i, g) = \mathcal{O}(y_i).$$

*In particular, the chain recurrence class $C(p)$ is wild.*

The proof of Corollary 1.2 is done by Theorem 1 and a standard argument involving Baire's category theorem, together with the upper semicontinuity of chain recurrence classes. We omit the proof.

As a direct corollary, we have the following:

**Corollary 1.3.** *On every closed 3-manifold $M$ there is a non-empty $C^1$-open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ such that every $f \in \mathcal{U}$ admits a hyperbolic periodic point of $s$-index 2 satisfying the following conditions:*

- *the chain recurrence class $C(p)$ is wild (and is a Hausdorff limit of a sequence of periodic orbits with trivial classes);*
- *the class $C(p)$ is volume hyperbolic;*
- *the class $C(p)$ is not contained in a normally hyperbolic surface.*

The argument involving the flexible periodic points in [BS] genuinely depended on the invariance of $C^1$-distance under rescaling. Thus the direct application of techniques developed in this paper does not work under the $C^r$-topology for $r \geq 2$. Consequently, we still do not have any answer to the following question.

**Question 1.** *Given $r \in [2, +\infty]$ and a 3-manifold $M$, does there exist $C^r$-locally generic diffeomorphisms on $M$ with a wild and volume hyperbolic chain recurrence class $C(p)$ which is not contained in a normally hyperbolic surface?*

## 1.3 Generic diffeomorphisms without attractors or repellers

Following [BLY] (see [BLY] for the bibliography of the results stated below without citation), we say that a compact invariant set $\Lambda \subset M$ is a *topological attractor* if the following holds:

- it is transitive;
- it has a compact attracting neighborhood $U$ (i.e., $U$ is a compact neighborhood of $\Lambda$ satisfying $f(U) \subset \mathrm{int}(U)$) such that $\bigcap_{i=0}^{+\infty} f^i(U) = \Lambda$ holds.

*Topological repellers* mean topological attractors for $f^{-1}$. Recall that a *quasi-attractor* is a chain recurrence class admitting a basis of attracting neighborhoods. By definition, an topological attractor is a quasi-attractor, but the converse is not true in general.

It is known that for $f \in \mathrm{Diff}^1(M)$, quasi-attractors always exist. In dimension 2, it is known that $C^1$-generic diffeomorphsism always have an attractor and a repeller. In dimension larger than

or equal to 4, [BLY] built $C^r$-generic diffeomorphisms, for any $r \geq 1$, without attractors or repellers. However, on 3-manifolds the argument in [BLY] was not enough to get generic diffeomorphisms without attractors or repellers: they only constructed $\mathcal{C}^r$-generic diffeomorphisms without attractors but with (infinitely many) repellers.

Our example, together with Theorem 1, allows to obtain generic existence of a quasi-attractor which is accumulated by saddles with trivial classes. As a result, we have the following:

**Corollary 1.4.** *On every closed 3-manifold $M$ there is a non-empty $C^1$-open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ and a residual subset $\mathcal{R} \subset \mathcal{U}$ such that every $f \in \mathcal{R}$ has no topological attractors or topological repellers. Moreover, such $\mathcal{U}$ and $\mathcal{R}$ can be chosen so that every $g \in \mathcal{R}$ has finitely many wild volume hyperbolic quasi-attractors whose basins cover a residual subset of $M$.*

We give a brief account on the proof of Corollary 1.4 in section 4.

Thus, for $C^1$-case, the problem of the existence of attractors and repellers is almost settled. However, again it seems that the direct application of this paper's techniques do not work under $C^r$-topology for $r \geq 2$. Therefore, the following is a remaining problem in this line:

**Question 2.** *Given $2 \leq r \leq +\infty$ and a 3-manifold $M$, does there exist $C^r$-locally generic diffeomorphisms on $M$ without attractors or repellers?*

**Organization of this paper** In section 2 we give the precise definition of partially hyperbolic filtrating Markov partitions and present some of the basic properties of them. In section 3, using these properties and the notion of $\varepsilon$-flexible points, we prove Theorem 1. In section 4, we prove Proposition 1.1: we construct examples of diffeomorphisms which satisfy the hypotheses of Theorem 1 and the volume hyperbolicity. We close Section 4 by giving a short explanation of Corollary 1.4.

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# 2 Partially hyperbolic filtrating Markov partitions

In this section we give the precise definition of partially hyperbolic filtrating Markov partition and discuss several elementary properties of it. Throughout this article, $M$ denotes a closed, smooth, three-dimensional Riemannian manifold. We denote the group of $C^1$-diffeomorphisms of $M$ by $\mathrm{Diff}^1(M)$ and furnish it with the $C^1$-topology.

## 2.1 Filtrating sets and chain recurrence classes

Let us start with recalling the notion of filtrating set. Let $f \in \mathrm{Diff}^1(M)$. A compact set $A \subset M$ is called *attracting region* of $f$ if $f(A) \subset \mathrm{int}(A)$, where $\mathrm{int}(A)$ denotes the topological interior of $A$. A compact set $R \subset M$ is a *repelling region* of $f$ if it is an attracting region of $f^{-1}$. Finally, a *filtrating set* $U$ of $f$ is a compact subset of $M$ which is an intersection of an attracting region $A$ and a repelling region $R$. The main property of a filtrating set $U$ is that any point $x \in U$ which goes out $C$ (i.e., $f(x) \notin U$) has no future return to $U$: for every $n > 0$ one has $f^n(x) \notin U$.

Let us fix a distance function $d$ on $M$ and let $\varepsilon > 0$. We say that a sequence of points $\{x_n\}_{n=0,\ldots,n_0}$ is an *$\varepsilon$-pseudo orbit* of $f$ if for every $k \in [0, n_0 - 1]$ we have $d(f(x_k), x_{k+1}) < \varepsilon$. The *chain recurrence set* $\mathcal{R}(f)$ of $f$ is the set of points $x$ such that for every $\varepsilon > 0$ there is an $\varepsilon$-pseudo orbit $x_0 = x, x_1, \ldots, x_{n_0} = x$ for some $n_0 \geq 1$. According to Conley theory, the chain recurrence set is naturally divided into a disjoint union of compact subsets called *chain recurrence classes* by the equivalence relation on $R(f)$ defined as follows: two points $x, y \in \mathcal{R}(f)$ belongs to the same chain recurrence class if for every $\varepsilon > 0$ there are two $\varepsilon$-pseudo orbits, one starting from $x$ and ending at $y$ and the other starting from $y$ and ending at $x$. One can see that the notion of chain recurrence and chain recurrence classes do not depend on the choice of $d$.

It is easy to check that every chain recurrence class $C$ meeting a filtrating set $U$ is contained in $U$. On the other hand, Conley theory implies that every chain recurrence class admits a basis of neighborhoods which are filtrating sets.

Finally, note that being a attracting region is a $C^0$ (and therefore $C^1$)-robust property: if $A$ is an attracting region for $f$, then $A$ is an attracting region for every $g$ sufficiently $C^0$-close to $f$. Similarly, being a repelling region is a $C^0$-robust property and consequently the same holds for filtrating sets.

## 2.2 Invariant cone fields, partial hyperbolicity and volume hyperbolicity

In this subsection we review the notion of cone fields and partial hyperbolicity.

Consider the cone $C = \{(x, y, z) \in \mathbb{R}^3, |z| \geq \sqrt{x^2 + y^2}\}$. In this paper a *cone* $\mathcal{C}_x$ at a point $x \in M$ means the image of $C$ by some linear isomorphisms from $\mathbb{R}^3$ to $T_xM$. A *(continuous) cone field* $\mathcal{C} = \{\mathcal{C}_x\}$ on a subset $U \subset M$ is a family of cones $\mathcal{C}_x$ of $T_xM$ ($x \in U$) which are locally defined as the image of $C$ by a linear isomorphisms $A_x \colon \mathbb{R}^3 \to T_xM$ depending continuously on $x$. In this article, we only treat continuous cone fields. Hence we omit the word continuous for cone fields.

For two cones $C_1$, $C_2$ at same point, we say that $C_1$ is *strictly contained* in $C_2$ if

$$C_1 \subset \mathrm{int}(C_2) \cup \{0\}.$$

In the same way, we say that a linear subspace $L$ is *strictly contained* in a cone $\mathcal{C}$ if $L \subset \mathrm{int}(\mathcal{C}) \cup \{0\}$.

**Definition 2.1.** *Let $U \subset M$. A cone field $\mathcal{C}$ on $U$ is said to be* strictly invariant *under $f$ if for every $x \in U$ with $f(x) \in U$ one has that $Df_x(\mathcal{C}_x)$ is strictly contained in $\mathcal{C}_{f(x)}$.*

*A cone field $\mathcal{C}$ is said to be* unstable cone field *if it is strictly invariant and there is a Riemann metric $\|\cdot\|$ on $M$ such that for every $x \in U$ with $f(x) \in U$, one has $\|Df(v)\| > \|v\|$ for every vector $v \in \mathcal{C}_x \setminus \{0\}$ .*

The notion of unstable cone field is closely related to the notion of partial hyperbolicity.

**Definition 2.2.** *Let $K \subset M$ be an $f$-invariant compact set. We say that $K$ is partially hyperbolic (with 1-dimensional unstable bundle) if there is a $Df$-invariant continuous splitting $TM|_K = E^{cs} \oplus E^u$ such that the following holds:*

- *The splitting is* dominated*: there is $n > 0$ such that for every $x \in K$, for every unit vectoru $\in E^{cs}(x)$ and $v \in E^u(x)$ we have the following inequality:*

$$\|Df^n(u)\| < \frac{1}{2}\|Df^n(v)\|.$$

  *By $E^{cs} \oplus_< E^u$ we mean that the bundle $E^{cs}$ is dominated by $E^u$.*

- *The bundle $E^u$ is* uniformly expanding*: there is $n > 0$ such that for every $x \in K$ and for every unit vector $v \in E^u(x)$, one has $\|Df^n(v)\| > 1$.*

In [Go], Gourmelon showed that, in the previous definition we can always choose a Riemannian metric for which $n = 1$. As a consequence we have the following:

**Lemma 2.3.** *Let $U \subset M$ be a compact subset and $K = \bigcap_{n \in \mathbb{Z}} f^n(U)$ be its maximal invariant set. Then,*

- *$K$ admits a dominated splitting $E \oplus_< F$ if and only if there is a strictly invariant cone field $\mathcal{C}$ on some neighborhood of $K$.*
- *$K$ is partially hyperbolic (with a 1-dimensional unstable bundle) if and only if there is a strictly invariant unstable cone field $\mathcal{C}$ on some neighborhood of $K$.*

Let $K$ be an invariant set with a dominated splitting $E \oplus_< F$ where $\dim E = 2$ and $\dim F = 1$. We say that $K$ is *volume hyperbolic* if the determinant of the derivative $Df$ restricted to $E$ is uniformly contracting and $Df$ restricted to $F$ is uniformly expanding. In this paper, for a linear isomorphism $L \colon V_1 \to V_2$ between two Euclidean spaces and a subspace $W_1 \subset V_1$, by *determinant of the restriction* $L|_{W_1}$ we mean the determinant calculated with respect to the Euclidean structure that $W_1$ and $L(W_1)$ inherit from $V_1$ and $V_2$, respectively.

The existence of a dominated, partially hyperbolic, or volume hyperbolic splitting on a maximal invariant set is a $C^1$-robust property in the following sense (see [BDV] for the detail).

**Lemma 2.4.** *Let $U$ be a compact subset of $M$, and $K = \bigcap_{n\in\mathbb{Z}} f^n(U)$ be its maximal invariant set. Suppose $K$ admits a dominated splitting $E \oplus_< F$. Then,*

- *if $g$ is sufficiently $C^1$-close to $f$, then the maximal invariant set $K_g := \bigcap_{n\in\mathbb{Z}} g^n(U)$ also admits a ($g$-invariant) dominated splitting $E_g \oplus_< F_g$, where $E_g$ and $F_g$ are the continuations of $E$ and $F$, respectively.*
- *if the splitting $E \oplus_< F$ is volume hyperbolic and $g$ is sufficiently $C^1$-close to $f$, then $E_g \oplus_< F_g$ is also volume hyperbolic.*

## 2.3 Partially hyperbolic filtrating Markov partitions

In this subsection, we give the precise definition of partially hyperbolic filtrating Markov partitions.

A compact subset $R$ of $M$ is said to be a *rectangle* if it is $C^1$-diffeomorphic to the full compact cylinder $\mathbb{D}^2 \times [0,1] \subset \mathbb{R}^3$, where $\mathbb{D}^2$ is the unit disc of $\mathbb{R}^2$. We endow $\mathbb{D}^2 \times [0,1]$ with the coordinates $(x, y, z)$ of $\mathbb{R}^3$.

We denote by $\partial_s R$ the image of $(\partial\mathbb{D}^2) \times [0,1]$ (which is diffeomorphic to the annulus $S^1 \times [0,1]$) and call it the *side boundary* of $R$. Also, we denote by $\partial_l R$ the image of $\mathbb{D}^2 \times \partial[0,1] = \mathbb{D}^2 \times \{0,1\}$ and call it the *lid boundary* of $R$ (which has two connected components). Given a rectangle $R \subset M$, a *vertical sub-rectangle* of $R$ is a rectangle $R_1 \subset R$ such that the following holds:

- $R_1$ is disjoint from $\partial_s(R)$, and
- $\partial_l(R_1) \subset \partial_l(R)$ and each connected component of $\partial_l(R)$ contains exactly one connected component of $\partial_l(R_1)$.

A *horizontal sub-rectangle* $R_2$ of $R$ is a rectangle $R_2 \subset R$ satisfying the following:

- $\partial_s(R_2) \subset \partial_s(R)$ and $\partial_s(R_2)$ is a essential sub-annulus of the annulus $\partial_s(R)$, and
- each connected component of $\partial_l(R_2)$ is either disjoint from $\partial_l(R)$ or coincides with a connected component of $\partial_l(R)$.

A cone field $\mathcal{C}$ on $R$ is called *vertical* if there is a diffeomorphism $\varphi\colon R \to \mathbb{D}^2 \times [0,1]$ such that for every $p \in R$ the cone $D\varphi(\mathcal{C}(p))$ contains the vertical vector $\frac{\partial}{\partial z}$ and is transverse to the horizontal plane field spanned by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

If $\mathcal{C}$ is a vertical cone field on $R$ and $R_1 \subset R$ is a vertical sub-rectangle, we say that $R_1$ is *$\mathcal{C}$-vertical* if the restriction of $\mathcal{C}$ to $R_1$ is a vertical cone field of $R_1$. Also, if $\mathcal{C}$ is a vertical cone field on $R$ and $R_1 \subset R$ is a horizontal sub-rectangle, we say that $R_1$ is *$\mathcal{C}$-horizontal* if the restriction of $\mathcal{C}$ to $R_1$ is a vertical cone field of $R_1$.

**Definition 2.5.** *Let $f \in \mathrm{Diff}^1(M)$. A compact subset $\mathbf{R}$ of $M$ is said to be a* partially hyperbolic filtrating Markov partition of saddle type of *$f$ if:*

- $\mathbf{R}$ *is a filtrating set: $\mathbf{R} = A \cap R$ where $A$ is an attracting region and $R$ is a repelling region.*
- $\mathbf{R}$ *is the union of finitely many pairwise disjoint rectangles $\mathbf{R} = \bigcup_{i=1,\dots,k} R_i$.*
- *The side boundary $\partial_s(R_i)$ is contained in $\partial A$ and the lid boundary $\partial_l(R_i)$ is contained in $\partial R$ (the boundary of repelling region).*
- *For every $(i, j)$, each connected component of $f(R_j) \cap R_i$ is a vertical sub-rectangle of $R_i$.*
- *There is a strictly invariant unstable cone field $\mathcal{C}^u$ on $\mathbf{R}$ which is vertical on each $R_i$.*

**Definition 2.6.** *Let $f \in \mathrm{Diff}^1(M)$. A compact subset $\mathbf{R}$ of $M$ is said to be a* partially hyperbolic filtrating Markov partition of attracting type of *$f$ if:*

- $\mathbf{R}$ *is an attracting region.*
- $\mathbf{R}$ *is the union of finitely many rectangles $\mathbf{R} = \bigcup_{i=1,\dots,k} R_i$.*

- *For every $i \neq j$, one of the following holds:*
  - *either $R_i$ and $R_j$ are disjoint,*
  - *or, $R_i \cap R_j$ is exactly one connected component of $\partial_l(R_i)$ or of $\partial_l(R_j)$ contained in the interior of a component of $\partial_l(R_j)$ or of $\partial_l(R_i)$ respectively. In this case we say that $R_i$ and $R_j$ are* adjacent.
- *The side boundary $\partial_s(R_i)$ is contained in $\partial A$.*
- *For every $(i, j)$, each connected component of $f(R_j) \cap R_i$ satisfies one of the following conditions:*
  - *Either it is a vertical sub-rectangle of $R_i$,*
  - *or, it is a connected component of $f(\partial_l(R_j))$ contained in a component of $\partial_l(R_i)$.*
- *There is a strictly invariant unstable cone field $\mathcal{C}^u$ on $\mathbf{R}$ which is vertical on each $R_i$.*

**Remark 2.7.** *In Definition 2.5, the fourth condition (on the shape of each connected components) can be derived from the other conditions. Meanwhile, the corresponding condition in Definition 2.6 cannot. Hence, for the sake of the consistency, we included this condition into the definition.*

**Remark 2.8.** *As the cone field $\mathcal{C}$ is strictly invariant, one can show that, in Definitions 2.5 and 2.6, if a connected component of $f(R_j) \cap R_i$ is a vertical sub-rectangle of $R_i$, then it is also $\mathcal{C}$-vertical.*

It would be possible that one gives a more general, conceptual definition of partially hyperbolic filtrating Markov partition including both types, but the definition and some proofs would become more technical. Since the main purpose of this article is to present examples, we prefer to stop pursuing the generality and continue the study of them.

In the rest of this section, we will investigate the basic properties of partially hyperbolic filtrating Markov partitions. Most of the proofs are quite classical. Hence we often avoid the formal proofs and only give the sketch of them. In many cases, the same proof works for the saddle type and the attracting type. However, the following lemma requires some modification depending on the types.

**Lemma 2.9.** *The property of being a partially hyperbolic filtrating Markov partition (of saddle or attracting type) is a $C^1$-robust property in the following sense:*

*(1) Assume that $\mathbf{R} = \bigcup R_i$ is a partially hyperbolic filtrating Markov partition of saddle type of a diffeomorphism $f$. Then there is a $C^1$-neighborhood $\mathcal{U}$ of $f$ such that for any $g \in \mathcal{U}$, $\mathbf{R}$ is a partially hyperbolic filtrating Markov partition of saddle type as well.*

*(2) Assume now that $\mathbf{R} = \bigcup R_i$ is a partially hyperbolic filtrating Markov partition of attracting type of a diffeomorphism $f$. Then there is a $C^1$-neighborhood $\mathcal{U}$ of $f$ so that, for any $g \in \mathcal{U}$, there is a diffeomorphism $\psi$ which is $C^1$-close to the identity such that $\mathbf{R}_g = \bigcup \psi(R_i)$ is a partially hyperbolic filtrating Markov partition of saddle type for $g$.*

*Sketch of the proof.* Let us start the proof for the saddle type case. Being a filtrating set is a $C^0$-robust property. Having a strictly invariant unstable cone field is a $C^1$-robust property and the fact that a cone field is vertical over a cylinder is also $C^1$-robust. Therefore the unique difficulty is to obtain the condition on the intersection $f(R_j) \cap R_i$.

The existence of the invariant cone field implies that $f(\partial_s(R_j))$ cuts transversely $\partial_l(R_i)$. Notice that $\partial_s(R_j)$ and $\partial_l(R_i)$ are compact surfaces with boundary and the fact that $f(\partial_s(R_j)) \subset \partial A$ and $\partial_l(R_i) \subset \partial R$ implies that this intersection $f(R_j) \cap R_i$ is disjoint from $\partial_s(R_i)$. Now one can deduce that, for $g$ which is sufficiently $C^1$-close to $f$, each connected component of $g(R_j) \cap R_i$ remains to be a $\mathcal{C}$-vertical sub-rectangle, since the side boundary of each connected component of $g(R_j) \cap R_i$ varies $C^1$-continuously.

For the attracting type, the union $S_f = \bigcup_i \partial_l(R_i)$ is a strictly positively $f$-invariant compact surface with boundary. A priori, this surface is not invariant for $g$ close to $f$. So we need to find a candidate for the lid boundaries of rectangles for $g$.

Note that the existence of the vertical strictly invariant unstable cone field $\mathcal{C}$ implies that $S_f$ is normally hyperbolic. Therefore, the persistence of normally hyperbolic invariant manifolds (see [HPS] for example) implies that $\partial_l R_i$ varies continuously under $C^1$-small perturbations of $f$.

Thus for $g$ which is $C^1$-close to $f$, we can construct cylinders $R_{i,g}$ whose boundary is close to the boundary of corresponding $R_i$ and so that $S_g = \bigcup \partial_l(R_{i,g})$ is a positively strictly invariant compact surface with boundary. Then, take the union $R_g = \bigcup R_{i,g}$. One can deduce that $R_g$ is an attracting region for $g$ and $\bigcup R_{i,g}$ is an attracting partially hyperbolic Markov partition for $g$. $\square$

## 2.4 Iteration, refinement of Markov partitions

Let $\mathbf{R} = \bigcup R_i$ be a partially hyperbolic filtrating Markov partition (of saddle or attracting type) of $f$ endowed with a vertical unstable cone field $\mathcal{C}$. Next proposition follows from standard arguments in hyperbolic dynamics, together with strongly invariant vertical unstable cones.

**Proposition 2.10.** *(1) Suppose that the connected component $f(R_i) \cap R_j$ is a vertical sub-rectangle of $R_j$. Then it is a $\mathcal{C}$-vertical and $Df(\mathcal{C})$-vertical sub-rectangle of $R_j$.*

*(2) Similarly, if the connected component of $f^{-1}(R_i) \cap R_j$ is a horizontal sub-rectangle of $R_j$, then it indeed is both $\mathcal{C}$-horizontal and $Df^{-1}(\mathcal{C})$-horizontal sub-rectangle of $R_j$.*

**Remark 2.11.** *The case where the connected component of $f(R_i) \cap R_j$ fails to be a vertical sub-rectangle happens only for attracting type. In such a case, the connected component is a $C^1$-disc contained in the interior of $\partial_l R_j$.*

*Proof of Proposition 2.10.* The proof of item (1) and (2) are similar so we only give the proof of the first one. Suppose that $R'$ is a connected component of $f(R_i) \cap R_j$ which is a vertical sub-rectangle of $R_j$. We need to find coordinates on $R'$ which guarantees that $R'$ is a $\mathcal{C}$-vertical sub-rectangle or $Df(\mathcal{C})$-vertical sub-rectangle. Let us find such coordinates.

Since $R_i$ is a $\mathcal{C}$-vertical rectangle, we can a the one-dimensional foliation $\mathcal{F}_i$ tangent to $\mathcal{C}$ on $R_i$ by taking the image of the $z$-direction. Also, since $R_j$ is a $\mathcal{C}$-vertical rectangle, we can take the two-dimensional foliation $\mathcal{H}_j$ which is transverse to $\mathcal{C}$ on $R_j$ by taking the image of the $xy$-plane.

Then, we consider the foliation $f(\mathcal{F}_i)$ and $\mathcal{H}_j$ on $R'$. Since $\mathcal{C}$ is $f$-invariant, we can see that $f(\mathcal{F}_i)$ and $\mathcal{H}_j$ are transverse at each point. Now we take a coordinate which sends $f(\mathcal{F}_i)$ parallel to the $z$-direction, $\mathcal{H}_j$ parallel to the $xy$-direction and $R'$ to $\mathbb{D}^2 \times I$. In this coordinates, one can check that $Df(\mathcal{C})$ and (in particular) $\mathcal{C}$ contain the $z$-direction and $\mathcal{C}$ (and therefore $Df(\mathcal{C})$) does not contain any vector in the $xy$-direction. This concludes that the sub-rectangle $R'$ is both $\mathcal{C}$- and $Df(\mathcal{C})$-vertical. $\square$

By a similar argument and taking the attracting case into consideration, we can prove the following:

**Proposition 2.12.** *(1) Let $\tilde{R}_i \subset R_i$ be a $\mathcal{C}$-vertical sub-rectangle. Then for every $j$, each connected component $\Gamma$ of $f(R_i) \cap R_j$ contains exactly one connected component $\tilde{\Gamma}$ of $f(\tilde{R}_i) \cap R_j$.*

*Notice that $\Gamma$ is either a $\mathcal{C}$-vertical sub-rectangle of $R_j$ or is contained in the interior of the lid boundary $\partial_l(R_i)$. In the first case, $\tilde{\Gamma}$ is $\mathcal{C}$- and $Df(\mathcal{C})$-vertical sub-rectangle of $R_j$.*

*(2) Let $\hat{R}_j \subset R_j$ be a $\mathcal{C}$-horizontal sub-rectangle. Then for every $i$, each connected component $\Delta$ of $f^{-1}(R_j) \cap R_i$ contains exactly one connected component $\hat{\Delta}$ of $f^{-1}(\tilde{R}_j) \cap R_i$. Furthermore, if $\Delta$ is a horizontal sub-rectangle of $R_j$ then $\hat{\Delta}$ is a $\mathcal{C}$- and $Df^{-1}(\mathcal{C})$-horizontal sub-rectangle of $R_j$.*

We omit the proof of Proposition 2.12.

If $\mathbf{R} = \bigcup R_i$ is a partially hyperbolic filtrating Markov partition of $f$, we denote by $\{\mathbf{R} \cap f(\mathbf{R})\}$ the family of connected components of $f(R_i) \cap R_j$ which are vertical sub-rectangles (i.e., those which have non-empty interior).

As a direct corollary of Proposition 2.12, we have the following:

**Corollary 2.13.** *Let $\mathbf{R} = \bigcup R_i$ be a partially hyperbolic filtrating Markov partition endowed with the vertical unstable cone field $\mathcal{C}$. Then $\mathbf{R} \cap f(\mathbf{R})$ is a partially hyperbolic filtrating Markov partition (of the same type) for the family of rectangles $\{\mathbf{R} \cap f(\mathbf{R})\}$. The cone fields $\mathcal{C}$ and $Df(\mathcal{C})$ are strictly invariant vertical unstable cone field on $\mathbf{R} \cap f(\mathbf{R})$.*

*Similarly, $\mathbf{R} \cap f^{-1}(\mathbf{R})$ is a partially hyperbolic filtrating Markov partition (of the same type) with the family of rectangles $\{\mathbf{R} \cap f^{-1}(\mathbf{R})\}$. The cone fields $\mathcal{C}$ and $Df^{-1}(\mathcal{C})$ are strictly invariant, vertical, unstable cone field on it.*

Iterating this process, we also have the following.

**Corollary 2.14.** *For any integers $m \leq n$, the intersection $\bigcap_{i=m}^{n} f^i(\mathbf{R})$ is a partially filtrating Markov partition with the family of rectangles $\{\bigcap_{i=m}^{n} f^i(\mathbf{R})\}$. The cone field $Df^i(\mathcal{C})$ is strictly invariant, vertical, unstable cone field on it for every $i$ satisfying $m \leq i \leq n$.*

For any integers $m \leq n$, the partially hyperbolic filtrating Markov partition $\bigcap_{i=m}^{n} f^i(\mathbf{R})$ is called a *refinement* of $\mathbf{R}$.

## 2.5 Center-stable discs and unstable lamination

For a partially hyperbolic filtrating Markov partition $\mathbf{R} = \bigcup R_i$ of $f$, we can associate two important dynamically defined sets, namely, the *unstable lamination* and the *center stable discs.* These sets have certain invariance under the iteration of $f$.

**Lemma 2.15.**

- *The positive maximal invariant set $\Lambda_+ = \bigcap_{n\geq 0} f^n(\mathbf{R})$ is a 1-dimensional lamination whose leaves intersect each rectangle $R_i$ as a continuous family of $C^1$-segments whose tangent space is contained in the cone field $Df^n(\mathcal{C}^u)$ at every point and for every $n \geq 0$.*
- *For the saddle type: the negative maximal invariant set $\Lambda_- = \bigcap_{n\leq 0} f^n(\mathbf{R})$ is a 2-dimensional lamination whose leaves are a continuous family of $C^1$-discs contained in the rectangles $R_i$, with boundary contained in $\partial_s(R_i)$ and transverse to each cone field $Df^n(\mathcal{C}^u)$. These discs are called* the center-stable discs.
- *For the attracting type: the partially hyperbolic filtrating Markov partition $\mathbf{R} = \bigcup R_i$ admits a unique positively invariant two-dimensional foliation satisfying the following properties:*
  - *each leaf is a 2-disc transverse to each cone field $Df^n(\mathcal{C}^u)$,*
  - *the connected components of $\bigcup_k \partial_l(R_k)$ are leaves of this foliation,*
  - *the family of discs induces on each $R_i$ a continuous family of $C^1$-discs,*
  - *the family of discs is upper semi continuous for the Hausdorff distance, the discontinuity being the $\partial_l R_k$.*

  *These discs are called* the center-stable discs.

*Sketch of the proof.* The existence of the one-dimensional unstable lamination is classical in partial hyperbolic dynamic. Let us build the center-stable discs which are the leaves of the center-stable foliation/lamination. For the sake of simplicity we only consider saddle type.

Let us fix one connected component $\Delta$ of $\bigcap_{n\leq 0} f^n(\mathbf{R})$. Using Proposition 2.12 repeatedly, we see that it is given as the limit of nested sequence of horizontal sub-rectangles $\{\tilde{R}_l\}$ where $\tilde{R}_l$ is the (unique) rectangle of the negative refinements $\bigcap_{i=-l}^{0} f^i(\mathbf{R})$ which contains $\Delta$.

As $f^{-1}$ contracts the vectors in the cone $Df(\mathcal{C}^u)$, one can check that the thickness of the rectangle tends to 0 as $l \to +\infty$. On the other hand, (the tangent space of) their lid boundary is contained in the complement cone of $Df^{-l}(\mathcal{C}^u)$, and this complement cone tends to a plane. Hence, we can see that such a decreasing sequence of such rectangles tends to a $C^1$-disc. The continuity of the family of discs follows by construction. $\square$

Using the notion of center-stable discs, we can define the *largeness* of stable manifolds of hyperbolic periodic points (of $s$-index 2) in a partially hyperbolic filtrating Markov partition.

**Definition 2.16.** *We say that a hyperbolic periodic point $x \in R_i$ of $s$-index 2 has large stable manifold if the center-stable disc through $x$ is contained in $W^s(x)$.*

Notice that for a hyperbolic periodic point of $s$-index 2 in $\mathbf{R}$, having large stable manifold is a $C^1$-robust property.

The next lemma shows that the property of having large stable manifold is a property of the orbit.

**Lemma 2.17.** *If $x \in \mathbf{R}$ is an $s$-index 2 hyperbolic periodic point with large stable manifold, then every point $f^i(x)$ in the orbit of $x$ has large stable manifold.*

*Proof.* Just notice that, for any center-stable disc $D$, each connected component of $f^{-1}(D) \cap \mathbf{R}$ is a center-stable disc. Then we can see that the stable disc through $f^{-1}(x)$ is contained in $W^s(f^{-1}(x))$. Now we can get the conclusion by a simple inductive argument. $\square$

The following lemma says that, in a hyperbolic basic set, the largeness of a periodic point is inherited to the periodic points passing nearby.

**Lemma 2.18.** *Let $K \subset \mathbf{R}$ be a hyperbolic basic set of $s$-index 2. Assume that $x \in K$ is a periodic point with large stable manifold. Then, there is a neighborhood $U$ of $x$ and a $C^1$-neighborhood $\mathcal{U}$ of $f$ such that the following holds: for every $g \in \mathcal{U}$ and every periodic point $y \in K_g$ whose orbit $\mathcal{O}(y, g)$ meets $U$, the stable manifold through $y$ is large.*

*Proof.* The lemma follows from the fact that local stable manifolds of points in a hyperbolic set vary continuously with the point and with the diffeomorphism. Therefore, for $g$ close to $f$ and $y \in K_g$ close enough to $x$, the stable manifold $W^s(y, g)$ contains the whole center-stable disc which passes through $y$. Since the property of having large stable manifold is invariant under the iteration (for periodic points), the property that the orbit of $y$ passes close enough to $x$ ensures the largeness of the stable manifold through $y$. $\square$

## 2.6 Topology of Markov partitions in the large stable manifolds and expulsion of periodic points from chain recurrence classes

Now we are ready to describe the "topology of a chain recurrence class" mentioned in the introduction. It plays an important role to determine the possibility of escaping from a chain recurrence class.

In this subsection, $D$ denotes a periodic center-stable disc of *period* $\pi(D)$ (that is, $D$ satisfies $f^{\pi(D)}(D) \subset D$ and $\pi(D)$ is the smallest positive integer satisfying this condition) in a partially hyperbolic filtrating Markov partition $\mathbf{R} = \bigcup R_i$ such that the first return map $f^{\pi(D)}|_D$ preserves the orientation. By $\lambda_D$ we denote the forward maximal invariant set in $D$, that is, we put $\lambda_D := \bigcap_{n \geq 0} f^{n\pi(D)}(D)$.

**Lemma 2.19.** *Under the above setting, consider the orbit space of $D \setminus \lambda_D$. Then it is naturally diffeomorphic to the two dimensional torus. We denote it by $\mathbb{T}_D$.*

*Sketch of the proof.* One can see this orbit space as follows: consider a fundamental domain of $f^{\pi(D)}$ which is a annulus bounded by a smooth curve $\gamma$ and $f^{\pi(D)}(\gamma)$. Then the quotient space is obtained from this annulus by gluing $\gamma$ to $f^{\pi(D)}(\gamma)$ along $f^{\pi(D)}$. $\square$

In this orbit space $\mathbb{T}_D$, there is a special (homotopy class of a) curve.

**Definition 2.20.** *The closed curve $\partial D$ induces a well-defined homotopy class in $\pi_1(\mathbb{T}_D)$. We call it the* parallel *of $\mathbb{T}_D$.*

Note that any curve in $D \setminus \lambda_D$ joining a point $y \in (D \setminus \lambda_D)$and $f^{\pi(D)}(y)$ projects to a closed curve in $\mathbb{T}_D$ whose (topological) intersection number with the parallel is equal to 1.

**Lemma 2.21.** *Under the above setting, $D \cap f^{\pi(D)}(\mathbf{R})$ consists of the union of finitely many pairwise disjoint discs, one of them being $f^{\pi(D)}(D)$.*

*Sketch of the Proof.* Let $R_D$ be a rectangle which contains $D$. For the saddle type $R_D$ is uniquely determined. For the attracting case there may be two such rectangles. We choose one of them. Then, by applying Proposition 2.12 repeatedly, we can see that each connected component of $f^{\pi(D)}(\mathbf{R}) \cap R_D$ is a $\mathcal{C}$-vertical sub-rectangle of $R_D$ or a disc contained in a lid boundary. As a result, we can see that the number of connected components of $D \cap f^{\pi(D)}(\mathbf{R})$ is finite and that each connected component is a $C^1$-disc in $D$. $\square$

We denote by $\Delta_D$ the projection of $\left(D \cap f^{\pi(D)}(\mathbf{R})\right) \setminus f^{\pi(D)}(D)$ to $\mathbb{T}_D$. It consists of finitely many pairwise disjoint discs embedded in the torus $\mathbb{T}_D$. Suppose that we have a hyperbolic periodic point $x \in \mathbf{R}$ of $s$-index 1. We are interested in deciding if the chain recurrence class of $x$ is trivial or not. To see this, it is important to compare the set $\Delta_D$ and the projection of the stable manifold of $x$ in $\mathbb{T}_D$.

First, let us have the following observation:

**Remark 2.22.** *Let $x \in D$ be a hyperbolic periodic point of period $\pi(D) = \pi(x)$ and with s-index 1. Assume that $W^s(x) \cap \lambda_D = \{x\}$. Then for the punctured stable manifold $W^s(x) \setminus \{x\}$ projected on $\mathbb{T}_D$ we have one of the following:*

- *Either it is a union of two disjoint simple closed curves such that for each curve the intersection number with the parallel is equal to 1 (this case happens when the stable eigenvalue of $x$ is positive);*
- *or, one simple closed curve whose intersection number with the parallel is 2 (this case happens when the stable eigenvalue of $x$ is negative).*

Lemma 2.21 shows the way for ensuring the triviality of a chain recurrence class of a periodic point in $\mathbf{R}$.

**Proposition 2.23.** *Assume that $x \in D$ is an s-index 1 hyperbolic periodic point with $\pi(x) = \pi(D)$ such that the projection of $W^s(x) \setminus \{x\}$ to $\mathbb{T}_D$ is a disjoint union of two simple closed curves having empty intersection with $\Delta_D$. Then the chain recurrence class of $x$ is trivial (i.e., is equal to the orbit of $x$).*

*Proof.* We fix an attracting region $A$ and a repelling region $R$ such that $\mathbf{R} = A \cap R$. Fix a disjoint union of two compact intervals $I = I_1 \cup I_2$ in $W^s(x) \setminus \{x\}$ which contains a fundamental domain for $f^{\pi(x)}$. It consists of two segments contained in different connected components of $W^s(x) \setminus \{x\}$.

**Claim 1.** *There is an integer $i$ such that $f^i(I) \cap A = \emptyset$.*

*Proof.* Since $A$ is an attracting region, for any $y \in M$, if $f^i(y) \notin A$ then $f^j(x) \notin A$ for every integers $i$ and $j$ satisfying $j < i$. By the compactness of $A$ and $I$, for proving the claim it is enough to prove that every $y \in I$ has an itinerary out of $A$.

For any point $y \in I$ we have an integer $j$ such that $f^j(y) \in D \setminus f^{\pi(D)}(D)$. By assumption, $f^j(y)$ does not belong to $f^{\pi(D)}(\mathbf{R})$. Meanwhile, since $x$ belongs to the interior of the repelling region $R$, $W^s(x)$ is contained in $R$. Now one deduces that $f^j(y) \notin f^{\pi(D)}(A)$. Accordingly we have $f^i(y) \notin A$ for $i \leq j - \pi(D)$. □

By the Claim 1 above, one can find a neighborhood $K$ of $I$ such that $f^i(K)$ is disjoint from $A$. As $x$ is a periodic point contained in $\mathbf{R}$, we have $x \in f^n(A) \subset \mathrm{int}(A)$ for every $n > 0$.

Suppose there exists a point $z \in \mathcal{R}(f) \setminus \mathcal{O}(x)$ (remember that $\mathcal{R}(f)$ denotes the chain recurrence set of $f$). By definition, for every $\varepsilon > 0$ we can find an $\varepsilon$-pseudo orbit $(y_k)_{k=0,\dots,n}$ such that $y_0 = y_n = x$ and $y_l = z$ for some $l$. Now, by using the fact that $x$ is a hyperbolic periodic point and $K$ contains a fundamental domain of $W^s(x)$ in its interior, one can deduce that if $\varepsilon$ is sufficiently small then the pseudo orbit $(y_k)$ must pass through $f^i(K)$.

However, notice that $K$ is disjoint from $A$. This contradicts the assumption that $y_0 = x \in f^{\pi(D)}(A) \subset \mathrm{int}(A)$: for $\varepsilon$ small enough an $\varepsilon$-pseudo orbit starting at $x$ cannot go outside $A$. □

## 2.7 Mixing Markov partitions and uniqueness of quasi-attractor

Given a partially hyperbolic filtrating Markov partition $\mathbf{R} = \bigcup R_i$ of $f$ we associate an *incidence matrix* which is a square matrix $A = (a_{ij})$ with integer entries, such that the term $a_{ij}$ is the number of connected components of $f(R_i) \cap R_j$ with non-empty interior.

**Definition 2.24.** *We say that a partially hyperbolic filtrating Markov partition $\mathbf{R} = \bigcup R_i$ is* transitive *if for every $(i, j)$ there exists an integer $n = n_{(i,j)} > 0$ such that $(i, j)$-component of $A^n$ is non-zero. Also, we say $\mathbf{R}$ is* mixing *if there exists $n > 0$ such that all the entries of $A^n$ are non-zero.*

We are interested in the transitivity or mixing property in regard to the uniqueness of the quasi-attractor in $\mathbf{R}$ for the attracting type. In this subsection, a *segment* $\sigma \subset M$ means an image of an embedding of an interval in $M$. A segment $\sigma \subset \mathbf{R}$ is $\mathcal{C}$*-vertical* if for every point $s \in \sigma$ we have $T_s\sigma \subset \mathcal{C}_s$, where $\mathcal{C}$ is a cone field on $\mathbf{R}$. A *complete vertical segment* is a segment contained in a rectangle $R_i$ such that the two end points connects the two different connected components of the lid boundary of $R_i$.

Let $\mathcal{C}$ be a strictly invariant unstable cone field over $\mathbf{R}$. If $\sigma$ is a $\mathcal{C}$-vertical segment in $\mathbf{R}$ satisfying $f(\sigma) \subset \mathbf{R}$, then by using the uniform expansion property of $f$ on $\mathcal{C}$, we can see that the length of $f(\sigma)$ is greater than that of $\sigma$ multiplied by some constant strictly greater than 1. Using this property, one can deduce the following:

**Lemma 2.25.** *Assume that $\mathbf{R} = \bigcup R_i$ is a mixing partially hyperbolic filtrating Markov partition of attracting type. Let $\mathcal{C}$ be a strictly invariant unstable cone field over it. Then given a segment $\sigma$ which is $\mathcal{C}$-vertical, there is $n_0 > 0$ such that for any $n \geq n_0$, the set $f^n(\sigma) \cap R_i$ contains a complete vertical segment for every $i$.*

Remember that a *homoclinic class* of a hyperbolic periodic point is the closure of the set of points of transversal intersection between the stable and the unstable manifold of it. As a direct corollary of Lemma 2.25, we have the following.

**Corollary 2.26.** *Suppose $\mathbf{R}$ is of attracting type and mixing, and $x \in \mathbf{R}$ is a hyperbolic periodic point of s-index 2 with large stable manifold. Then,*

- *The homoclinic class of $x$ is non-trivial.*
- *Every leaf of the unstable lamination of $\mathbf{R}$ cuts $W^s(x)$.*

Finally, we obtain the following:

**Corollary 2.27.** *Suppose that $\mathbf{R} = \bigcup R_i$ is a mixing partially hyperbolic filtrating Markov partition of attracting type and $x \in \mathbf{R}$ is a periodic point with large stable manifold. Then for every $g$ which is sufficiently $C^1$-close to $f$, $g$ admits a unique quasi-attractor in $\mathbf{R}$ and it is equal to the chain recurrence class of the continuation $x_g$ of $x$. Furthermore, the basin of this quasi-attractor is residual in $\mathbf{R}$.*

*Proof.* For the first claim, we just need to repeat the argument in [BLY]: any quasi-attractor contained in $\mathbf{R}$ is saturated by the strong unstable lamination. Since each leaf of the unstable lamination cuts the large stable manifold of $x$ (see Corollary 2.26), we deduce that $x$ belongs to the closure of the quasi-attractor, so it belongs to the quasi-attractor. This shows that every quasi-attractor in $\mathbf{R}$ is the chain recurrence class of $x$, which implies its uniqueness. Since the largeness of the stable manifold for a periodic point and the existence of partially hyperbolic filtrating Markov partitions are both $C^1$-robust (see Lemma 2.9 for the second item), we have the uniqueness for $g$ sufficiently $C^1$-close to $f$.

Now let $U_n$ be a basis of neighborhood of the class of $x$ consisting of attracting regions. Let $O_n \subset \mathbf{R}$ be the basin of the attracting region $U_n$. As the (entire) stable manifold of $x$ cuts every vertical segment, one deduces that $O_n$ is a open dense subset of $\mathbf{R}$ for every $n$. Then $\bigcap_{n\geq 0} O_n$ is the basin of the quasi-attractor which is a residual subset of $\mathbf{R}$.

□

# 3 Ejection of flexible periodic points

In this section, we review some of the results of [BS]. Then, by using these results, we finish the proof of Theorem 1.

## 3.1 Flexible points: definition

In [BS] the authors defined the notion of *flexible periodic points* and proved their generic existence under certain partially hyperbolic setting which includes partially hyperbolic filtrating Markov partitions. Let us recall it.

A *(2-dimensional) periodic linear cocycle of period $n$* is a map $\mathcal{A}\colon (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{R}^2 \to (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{R}^2$ of the form $\mathcal{A}(i,p) = (i+1, A_i(p))$, where $A_i \in GL(2,\mathbb{R})$. For linear cocycles $\mathcal{A}$, $\mathcal{B}$ of period $n$, we define the *distance* between $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathrm{dist}(\mathcal{A},\mathcal{B})$, as follows:

$$\mathrm{dist}(\mathcal{A},\mathcal{B}) := \sup \|\mathcal{A}(x) - \mathcal{B}(x)\|,$$

where $x$ ranges over all unit vectors in all fibers, that is, all unit vectors in $\coprod_{i \in \mathbb{Z}/n\mathbb{Z}}(\{i\} \times \mathbb{R}^2)$. This defines a distance on the space of linear cocycles of period $n$.

Let $\mathcal{A}_t$ denote a continuous one-parameter family of cocycles, that is, a continuous map from an interval to the space of cocycles. We define the *diameter of the family* $\mathcal{A}_t$ as

$$\operatorname{diam}(\mathcal{A}_t) := \sup_{s<t} \operatorname{dist}(\mathcal{A}_s, \mathcal{A}_t).$$

To define the notion of flexible points, we first give the definition of *flexible cocycles*.

**Definition 3.1.** *Let $\mathcal{A} = \{A_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$, $A_i \in \mathrm{GL}(2, \mathbb{R})$ be a linear cocycle of period $n > 0$ and let $\varepsilon$ be a positive real number. We say that $\mathcal{A}$ is* $\varepsilon$-flexible *if there is a continuous one-parameter family of linear cocycles $\mathcal{A}_t = \{A_{i,t}\}_{i \in \mathbb{Z}/n\mathbb{Z}}$ defined for $t \in [-1, 1]$ such that the following holds:*

- $\operatorname{diam}(\mathcal{A}_t) < \varepsilon$*;*
- *$A_{i,0} = A_i$, for every $i \in \mathbb{Z}/n\mathbb{Z}$;*
- *the product matrix $A_{-1} := (A_{n-1,-1}) \cdots (A_{0,-1})$ is a homothety (remember that a square matrix is called* homothety *if it has the form $\lambda \,\mathrm{Id}$, where $\lambda$ is a positive constant);*
- *for every $t \in (-1, 1)$, the product matrix $A_t = (A_{n-1,t}) \cdots (A_{0,t})$ has two distinct positive contracting eigenvalues;*
- *let $\lambda_t$ denote the smallest eigenvalue of the product matrix $A_t$. Then $\max_{-1 \leq t \leq 1} \lambda_t < 1$;*
- *$A_1$ has a real positive eigenvalue equal to $1$.*

Now let us see the definition of flexible periodic points.

**Definition 3.2.** *Let $M$ be a closed $3$-manifold, $f \in \mathrm{Diff}^1(M)$ and $\mathbf{R} = \bigcup R_i$ be a partially hyperbolic filtrating Markov partition of $f$ (of saddle or attracting type). A periodic orbit $\mathcal{O}(x) \subset \mathbf{R}$ is called* $\varepsilon$-flexible *if the two dimensional linear cocycle of the restriction of the derivatives to the center-stable bundle along its orbit is $\varepsilon$-flexible. A periodic point is called $\varepsilon$-flexible if its orbit is so.*

**Remark 3.3.** *To define the notion of flexibility of periodic points, we do not need the notion of partially hyperbolic filtrating Markov partitions (partially hyperbolic structure is enough). Since in this paper we only treat flexible points appearing in partially hyperbolic filtrating Markov partitions, we adopted the definition above.*

## 3.2 Isolation of flexible points

In this subsection, we state a perturbation result on the isolation of flexible periodic points (see Proposition 3.4).

In [BS], the authors showed that an $\varepsilon$-flexible periodic point has great freedom of changing the position of their (strong) stable manifold by an $\varepsilon$-perturbation which is supported in an arbitrarily small neighborhood of the orbit. By using such flexibility of flexible points (whose precise meaning will be elucidated in the next subsection), we can prove the following:

**Proposition 3.4.** *Let $\mathbf{R} = \bigcup R_i$ be a partially hyperbolic filtrating Markov partition (of saddle or attracting type) of a diffeomorphism $f$. Let $\varepsilon_0$ be a positive real number such that $\mathbf{R} = \bigcup R_i$ is still a partially hyperbolic filtrating Markov partition for every $g$ which is $\varepsilon_0$-$C^1$-close to $f$ (see Lemma 2.9).*

*Fix a positive real number $\varepsilon$ satisfying $0 < \varepsilon < \varepsilon_0$. Suppose that $x \in \mathbf{R}$ is a hyperbolic periodic point of $s$-index $2$ with large stable manifold and $\varepsilon$-flexible. Then, there is an $\varepsilon$-$C^1$-perturbation $g$ of $f$, supported in an arbitrarily small neighborhood of the orbit of $x$ such that the following holds:*

- *$f$ and $g$ coincide on the orbit of $x$. In particular, $x$ is a periodic point for $g$ with the same orbit.*
- *$x$ is an $s$-index $1$ hyperbolic periodic point of $g$, having an unstable eigenvalue arbitrarily close to $1$.*
- *the chain recurrence class of the orbit of $x$ is trivial for $g$.*

We will see how we deduce Proposition 3.4 from the results of [BS] in the rest of this section. As an immediate corollary of this proposition, we have the following:

**Corollary 3.5.** *Under the hypotheses of Proposition 3.4, an $\varepsilon$-perturbation $h$ of $g$ turns $x$ into an $s$-index 2 hyperbolic periodic point with trivial chain recurrence class.*

*Proof.* Proposition 3.4 produces a diffeomorphism $g$ for which $x$ is an $s$-index 1 hyperbolic periodic point of $g$, having an unstable eigenvalue arbitrarily close to 1 and a trivial chain recurrence class. This implies that the orbit of $x$ admits a filtrating neighborhood $U$ in which the orbit of $x$ is the maximal invariant set. Since $x$ has an unstable eigenvalue very close to 1, there is a $C^1$-small perturbation, supported in an arbitrarily small neighborhood of $x$ (hence in $U$), such that the maximal invariant set in $U$ is the orbit of a periodic segment on which the chain recurrent set consists of 3 periodic orbits, 2 of them being $s$-index 1 and the rest is the orbit of $x$ (which has $s$-index 2). $\square$

The proof of Proposition 3.4 will be done by realizing a perturbation given by [BS, Theorem 1.1] and [BS, Corollary 1.1] in a 2-dimensional setting, which is an extraction of the dynamics in the center-stable disc passing through $x$. Indeed, the proof of Proposition 3.4 is an almost immediate corollary of Corollary 1.2 of [BS]. Because of the importance of this step and for the convenience of the reader, we will present the argument here again.

## 3.3 Background from [BS]: the two dimensional setting

Now we cite Theorem 1.1 and Corollary 1.1 of [BS], which is used in the proof of Proposition 3.4.

To state them, let us consider the following general setting. Let $S$ be a surface (a smooth two dimensional manifold, which may have non-empty boundary and may be non-connected) and $F\colon S \to S$ be a $C^1$-diffeomorphisms on its image. Let $\varepsilon > 0$ and $x \in S$ be a periodic point of $F$ with period $\pi(x)$. Suppose that $x$ is an $\varepsilon$-flexible (attracting) periodic point, that is, the differential $DF$ along the orbit of $x$ is an $\varepsilon$-flexible cocycle.

Let $D$ be an attracting periodic disc of period $\pi(x)$, that is, $D, F(D), \dots, F^{\pi(x)-1}(D)$ are pairwise disjoint and $F^{\pi(x)}(D)$ is contained in the interior of $D$. We assume that $D$ is contained in the local stable manifold of $x$. As in Lemma 2.19, the orbit space of $F^{\pi(x)}$ in $D \setminus \{x\}$ is diffeomorphic to the torus $\mathbb{T}_D$. In the following, we denote this orbit space by $\mathbb{T}_F$. Then, $\partial D$ induces a homotopy class called parallel and the strong stable separatrices of $x$ induce two homotopy classes of curves on the orbit space $\mathbb{T}_F$ (recall that the flexibility assumption implies that $x$ has two distinct positive contracting eigenvalues). We call this homotopy class *meridian*.

We consider perturbations $G$ of $F$ preserving the orbit of $x$, supported in a small neighborhoods of $\mathcal{O}(x)$ such that $G^{\pi(x)}$ coincides with $F^{\pi(x)}$ on $D \setminus F^{\pi(x)}(D)$. By $\Lambda_G$ we denote the maximal invariant set of $G^{\pi(x)}$ in $D$ (i.e., $\Lambda_G := \bigcap_{n\geq 0} G^{n\pi(x)}(D)$) and by $\mathbb{T}_G$ we denote the orbit space of $G^{\pi(x)}$ on $D\setminus\Lambda_G$. One can see that $\mathbb{T}_G$ is diffeomorphic to the torus as well. Then, for $G$ satisfying these conditions, we identify $\mathbb{T}_F$ and $\mathbb{T}_G$ through the (unique) conjugacy between $F^{\pi(x)}|_{D\setminus\{x\}}$ and $G^{\pi(x)}|_{D\setminus\Lambda_G}$ which is the identity map on $D \setminus F^{\pi(x)}(D) = D \setminus G^{\pi(x)}(D)$.

Under this identification, the freedom of flexible points can be formulated by the following Theorem 2, which is proved in [BS, Theorem 1.1]:

**Theorem 2.** *Let $F$ be a $C^1$-diffeomorphism of a surface $S$, $x$ an $\varepsilon$-flexible periodic point and $D$ an attracting periodic disc of period $\pi(x)$. Assume that $D$ contains $x$ and is contained in the stable manifold of $x$. Let $\gamma = \gamma_1 \cup \gamma_2 \subset \mathbb{T}_F$ be the two simple closed curves which $W^{ss}(x)$ projects to.*

*Then, for any pair of $C^1$-curves $\sigma = \sigma_1 \cup \sigma_2$ embedded in $\mathbb{T}_F$ which is isotopic to $\gamma_1 \cup \gamma_2$, there is an $\varepsilon$-perturbation $G$ of $F$, supported in an arbitrarily small neighborhood of the orbit of $p$ such that $G$ satisfies the following:*

- *$G$ coincides with $F$ along the orbit of $x$ (in particular $x$ is a periodic point of $G$ with the same period $\pi(x)$);*
- *$x$ is a (non-hyperbolic) periodic attracting point having an eigenvalue $\lambda_1 \in ]0, 1[$ and an eigenvalue $\lambda_2 = 1$;*
- *$D$ is contained in the basin of $x$;*

- *the strong stable manifold $W^{ss}(x, G)$ projects to $(\sigma_1 \cup \sigma_2) \subset \mathbb{T}_G \simeq \mathbb{T}_F$.*

**Remark 3.6.** *In [BS, Theorem 1.1], the theorem is stated for $C^1$-diffeomorphism of surfaces. Since the perturbation we obtain is a local one, as we stated above, the same result is also true for local diffeomorphisms (diffeomorphisms on their images).*

The orbit of $x$ for $G$ is a non-hyperbolic attracting periodic point, having an eigenvalue equal to 1. By an extra, arbitrarily small perturbation, we can change the index of $x$ such that the strong stable manifold becomes the new stable manifold. Therefore we have the following (see Corollary 1.1 in [BS]).

**Corollary 3.7.** *Under the hypotheses of Theorem 2, there is an $\varepsilon$-perturbation $H$ of $F$, supported in an arbitrarily small neighborhood of $x$ such that the following holds:*

- *$x$ is a periodic saddle point having two real eigenvalues $0 < \lambda_1 < 1 < \lambda_2$, with $\lambda_2$ arbitrarily close to 1;*
- *$W^s(x) \setminus \mathcal{O}(x)$ is disjoint from the maximal invariant set $\Lambda_H$;*
- *$W^s(x, H)$ projects to $(\sigma_1 \cup \sigma_2) \subset \mathbb{T}_H \simeq \mathbb{T}_F$.*

## 3.4 Ejecting a flexible periodic point: proof of Proposition 3.4

Now, by Corollary 3.7 (which is proved by Theorem 2 and), let us finish the proof of Proposition 3.4.

*Proof of Proposition 3.4.* Let $\mathbf{R} = \bigcup R_i$ be a partially hyperbolic Markov partition of a diffeomorphism $f$ and $x$ an $\varepsilon$-flexible periodic point of period $\pi(x)$ with large stable manifold. Let $D$ be the center-stable disc containing $x$. Note that by Lemma 2.17, each point $f^i(x)$ also has large stable manifold for every $i$.

According to Lemma 2.21, the intersection $D \cap f^{\pi(x)}(\mathbf{R})$ is a union of finite number of disjoint discs, one of them being $f^{\pi(x)}(D)$. The projection $\Delta_D \subset$ of $(D \cap f^{\pi(x)}(\mathbf{R})) \setminus f^{\pi(x)}(D)$ to the torus $\mathbb{T}_D$ is a union of pairwise disjoint $C^1$-discs. Thus every homotopy class of simple closed curves in $\mathbb{T}_D$ contains curves disjoint from this union. We fix two disjoint simple curves $\sigma_1, \sigma_2 \subset \mathbb{T}_D$ disjoint from $\Delta_D$ and isotopic to the meridian (i.e., the projection of the strong stable separatrices of $x$).

We apply Corollary 3.7 to the restriction $F$ of $f$ to the surface $S$ which is the union of center-stable discs passing through points $f^i(x)$, $i \in \{0, \dots, \pi(x)-1\}$ (notice these discs are mutually distinct, because each $f^i(x)$ has large stable manifold). We obtain an $\varepsilon$-perturbation $H$ of $F$ satisfying the following conditions:

- $H$ coincides with $F$ on the orbit of $x$ and out of an arbitrarily small neighborhood of the orbit of $x$ (in particular $H^{\pi(x)}$ coincides with $F^{\pi(x)}$ on $D \setminus F^{\pi(x)}(D)$ and as a result the orbit space $\mathbb{T}_H$ is identified with $\mathbb{T}_F$, as explained in the last subsection);
- $x$ is a saddle fixed point of $H^{\pi(x)}$ whose stable separatrices are disjoint from the maximal invariant set of $H^{\pi(x)}$ in $D$ and their projections on $\mathbb{T}_F = \mathbb{T}_H$ are $\sigma_1$ and $\sigma_2$.

Then we realize $H$ as an $\varepsilon$-perturbation $h$ of $f$, which coincides with $H$ on the surface $S$ and with $f$ out of an arbitrarily small neighborhood of the orbit of $x$. We take $H$ with its support sufficiently small such that $h^{\pi(x)}(\mathbf{R}) = f^{\pi(x)}(\mathbf{R})$ holds. By construction, the stable separatrices of $x$ for $h$ are disjoint from $(D \cap h^{\pi(x)}(\mathbf{R})) \setminus h^{\pi(x)}(D)$ in the fundamental domain $D \setminus H^{\pi(x)}(D) = D \setminus F^{\pi(x)}(D)$.

By the choice of $\varepsilon$, the compact set $\mathbf{R}$ is still a partially hyperbolic filtrating Markov partition for $h$ and the point $x \in D$ is a hyperbolic saddle of $s$-index 1 of $h$ whose stable separatrices are disjoint from $(D \cap h^{\pi}(\mathbf{R})) \setminus h^{\pi}(D)$. Now Proposition 2.23 implies that the chain recurrence class of the orbit of $x$ is trivial, which concludes the proof. $\square$

## 3.5 Existence of flexible points with large stable manifolds

In order to prove Theorem 1 by Proposition 3.4 and Corollary 3.5, the only thing we need to prove is the existence of flexible points with large stable manifold under the assumption of Theorem 1. We discuss it in this subsection.

**Lemma 3.8.** *Let $\mathbf{R} = \bigcup R_i$ be a partially hyperbolic filtrating Markov partition of a diffeormophism $f$ and $\mathcal{U}$ a $C^1$-neighborhood of $f$ in which one can find a continuation of $\mathbf{R}$ (see Lemma 2.9). Assume that there are hyperbolic periodic points $p, p_1, q \in \mathbf{R}$ varying continuously with respect to $f \in \mathcal{U}$ such that for every $g \in \mathcal{U}$ they satisfy the following:*

- *$p_g$ is of s-index 2 and has large stable manifold;*
- *$p_{1,g}$ is of s-index 2 and has complex (non-real) stable eigenvalues homoclinically related with $p$;*
- *$q_g$ is of s-index 1 and $C(p_g) = C(q_g)$.*

*Then, for any $\varepsilon > 0$, there is a $C^1$-open and dense subset $\mathcal{D}$ of $\mathcal{U}$ such that every diffeomorphism $f \in \mathcal{D}$ has a periodic point $x \in C(p)$ of s-index 2, with large stable manifold, $\varepsilon$-flexible, homoclinically related with $p$, and whose orbit is $\varepsilon$-dense in $C(p)$.*

*Proof.* First notice that, if $x$ satisfies the announced properties for some $0 < \varepsilon' < \varepsilon$ and a diffeomorphisms $f \in \mathcal{U}$ then there is a $C^1$-neighborhood of $f$ such that the continuation of $x$ satisfies the announced properties for the same $\varepsilon$. Thus, it is enough to prove the $C^1$-density of $f$ (in $\mathcal{D}$) having a periodic point satisfying the properties as claimed.

Remember that [BS, Theorem 2] already announced the generic existence of flexible periodic points whose orbits are $\varepsilon$-dense in the chain recurrence class of $p$. The novelty here is that we require these flexible points have large stable manifolds. Therefore we cannot apply directly the statement of [BS, Theorem 2], and we need to go back to two important steps of its proof.

First, [BS, Proposition 6] (essentially based on [ABCDW]) asserts that any $f \in \mathcal{U}$ is $C^1$-approximated by a diffeomorphism $g$ for which $p$ is homoclinically related with a periodic point $y$ with the following properties:

- $y$ has a stable eigenvalue arbitrarily close to 1;
- the smallest Lyapunov exponent of $y$ is strictly smaller than a given negative constant $\lambda_f < 0$ (which only depends on $f$);
- the orbit of $y$ is arbitrarily close (say $\varepsilon/3$-close) to the chain recurrence class $C(p)$ in the Hausdorff distance.

Then, let us consider a hyperbolic basic set $\Lambda$ containing $y$ and (the continuations of) $p$ and $p_1$. As the orbit of $y$ is $\varepsilon/3$-dense in $C(p)$, it also holds for the hyperbolic set $\Lambda$. By the upper semicontinuity of the chain recurrence class, this $\varepsilon/3$-density in $C(p)$ persists under small perturbations.

This hyperbolic basic set $\Lambda$ has the stable bundle of dimension 2. Since $p_1$ has non-real stable eigenvalues, this stable bundle does not admit any dominated splitting. Now [BS, Proposition 7] asserts that $\Lambda$ admits arbitrarily small perturbations such that the continuation of $\Lambda$ has $\varepsilon$-flexible points $x_n$ whose orbits are arbitrarily close to $\Lambda$ in the Hausdorff distance and in particular are $\varepsilon$-dense in $C(p)$.

Now we can complete the proof by Lemma 2.18: the orbit of $x_n$ passes through a small neighborhood of $p$ which has large stable manifold. Therefore Lemma 2.18 ensures that the orbit of $x_n$ has a point with large stable manifold, and Lemma 2.17 ensures that this property holds for every point in the orbit of $x_n$. □

## 3.6 End of the proof of Theorem 1

Now let us finish the proof of Theorem 1 by Proposition 3.4, Corollary 3.5 and Lemma 3.8.

*Proof of Theorem 1.* We consider a diffeomorphism $f$ with a filtrating partially hyperbolic Markov partition $\mathbf{R} = \bigcup R_i$ of either saddle type or attracting type such that there are:

- an $s$-index 2 periodic point $p \in \mathbf{R}$ with large stable manifold,
- an $s$-index 2 periodic point $p_1$ homoclinically related with $p$ having a complex (non-real) stable eigenvalue, and
- a periodic point $q$ of $s$-index 1 which is $C^1$-robustly in the chain recurrence class $C(p, f)$.

We apply Lemma 3.8 to $f$: for any $\varepsilon > 0$, an arbitrarily small perturbation produces $\varepsilon$-flexible points homoclinically related with $p_1$ (hence with $p$), having large stable manifold, and whose orbit is $\varepsilon$ dense in $C(p_1) = C(p)$.

Then Proposition 3.4 and Corollary 3.5 allow us to create the announced points $x_g$ and $y_g$ (of $s$-index 1 and 2-respectively) with trivial chain recurrence classes and whose orbits are $2\varepsilon$ close to $C(p, f)$ with respect to the Hausdorff distance. □

# 4 Examples

In this section, we prove Proposition 1.1, that is, we show how we construct examples which satisfy the hypothesis of Theorem 1 with volume hyperbolicity. We also briefly discuss the proof of Corollary 1.4 at the end of this section.

## 4.1 Outline of the construction

The construction is done by the following two Propositions.

**Proposition 4.1.** *For every closed* 3*-manifold* $M$ *there exists a diffeomorphism* $f$ *of* $M$ *having a mixing hyperbolic filtrating Markov partition* $\mathbf{R} = \bigcup R_i$ *(of saddle type or attracting type) such that the following holds:*

- $\mathbf{R}$ *contains a non-trivial homoclinic class* $H(p)$ *where* $p$ *is a hyperbolic fixed point of* $f$ *in* $\mathbf{R}$*;*
- $p$ *has large stable manifold.*

*Such examples can be taken so that the maximal invariant set in* $\mathbf{R}$ *is a transitive hyperbolic set of* $s$*-index two. In particular, it admits a uniformly hyperbolic splitting* $E^s \oplus E^u$ *with* $\dim(E^s) = 2$.

**Remark 4.2.** *Indeed, for proving Proposition 4.1, we will construct an attracting ball* $B^3$ *containing the announced mixing hyperbolic filtrating Markov partition: then we embed it into any* 3*-manifold. Furthermore, for the attracting type, we can take the Markov partition in such a way that the whole ball is contained in the basin of attraction of it.*

To state the second proposition, recall that two hyperbolic periodic points $p_1, p_2$ of a diffeomorphism is said to have a *heterodimentional cycle* if they have different indices and $W^s(p_1) \cap W^u(p_2)$ and $W^s(p_2) \cap W^u(p_1)$ are both non-empty.

**Proposition 4.3.** *Let* $\mathbf{R} = \bigcup R_i$ *be a partially hyperbolic filtrating Markov partition of saddle type or attracting type satisfying the conclusion of Proposition 4.1. Then, we can find a diffeomorphism* $g$ *which also satisfies two conditions in the conclusion of Proposition 4.1 (with the same* $\mathbf{R} = \bigcup R_i$ *being a partially hyperbolic filtrating Markov partition) and the following conditions:*

*(1) There exists a hyperbolic periodic point* $p_1$ *homoclinically related with* $p$ *such that* $Dg^{\pi(p_1)}(p_1)|_{E^s}$ *has two contracting complex (non-real) eigenvalues (where* $\pi(p_1)$ *denotes the period of* $p_1$*).*

*(2) There exists a hyperbolic periodic point* $q$ *of* $s$*-index* 1 *which has a heterodimensional cycle associated with* $p$*.*

*Furthermore,* $g$ *can be taken so that its maximal invariant set in* $\mathbf{R}$ *is volume hyperbolic.*

Let us first see how these two propositions conclude the construction of the examples. In the construction we use the following lemma from [BDK] (see Theorem 1 of [BDK]).

**Lemma 4.4.** *Let $f$ be a diffeomorphism having a heterodimensional cycle between two hyperbolic periodic points whose difference of stable indices is one. If one of them has non-trivial homoclinic class, then, by $C^1$-arbitrarily small perturbation, we can find a diffeomorphism such that (the continuations of) two periodic points belong to the same chain recurrence class $C^1$-robustly.*

*Proof of Proposition 1.1.* First, by Proposition 4.1, for any 3-manifold we take a diffeomorphism $f$ with a partially hyperbolic (indeed, uniformly hyperbolic) filtrating Markov partition $\mathbf{R} = \bigcup R_i$ either saddle or attracting type satisfying the conclusion of Proposition 4.1. Then, by applying Proposition 4.3 to $f$, we obtain $g$ having a hyperbolic periodic point $p_1$ homoclinically related with $p = p_g$ and having complex eigenvalues, and a hyperbolic periodic point $q$ that has a heterodimensional cycle with $p = p_g$. Then, by applying Lemma 4.4 to the heterodimentional cycle between $p_{g_2}$ and $q$, we get a diffeomorphism which satisfies all the previous properties and furthermore $q$ and $p$ being robustly in the same chain recurrence class. This gives the announced diffeomorphism. □

## 4.2 Example of filtrating (partially) hyperbolic Markov partition: proof of Proposition 4.1

The proof of Proposition 4.1 follows by well known arguments, so we only give the sketch of it. In the following, as announced in Remark 4.2, we construct attracting endomorphisms (which are diffeomorphisms to their images) on three dimensional ball $B^3$ to itself containing the partially hyperbolic filtrating Markov partition. The construction on given manifold can be done by extending it as a diffeomorphism.

For the construction of the example of saddle type, we take a structurally stable diffeomorphisms on the sphere $S^2$ whose non-wandering set consists of the Smale's horseshoe, one source and one sink. By multiplying it by a contraction on the interval, one gets an attracting region diffeomorphic to $S^2 \times [-1, 1]$, which can be seen as a neighborhood of the boundary of the ball $B^3$. One can complete the dynamics on $B^3$ by adding a source. Note that one can embed this dynamics on any 3-manifold. This gives us a mixing (hyperbolic) filtrating Markov partition of saddle type.

For the attracting type, we take the Smale's solenoid attractor defined on a solid torus. It is not difficult to take such a diffeomorphism on $B^3$ so that $B^3$ is contained in the basin of attraction of the solid torus (see the construction in [BLY] based on [Gi]).

To complete the construction of attracting type, we need an additional argument on the shape of rectangles. Remember that we require that for any non-empty intersection of two rectangles the lid boundary of one of the rectangles is contained in the interior of that of the other. In order to get this property we make a tricky choice of rectangles as follows: suppose $F : S^1 \times \mathbb{D}^2 \to S^1 \times \mathbb{D}^2$, $F(\theta, x) = (2\theta, F_\theta(x))$ is a Smale's solenoid map. Then we divide $S^1 \times \mathbb{D}^2$ into four pieces $P_i := \{(\theta + (i/4), x) \mid \theta \in [0, 1/4], x \in \mathbb{D}^2\}$, $i = 0, 1, 2, 3$. Then we shrink $P_1$ and $P_3$ a little bit in $\mathbb{D}^2$ direction. Then the family of rectangles $\{P_i\}$ gives the desired example.

**Remark 4.5.** *In the argument above, we started from the Smale's solenoid attractor. Note that the same construction works for any Williams type attractor (see [W]).*

## 4.3 Complex eigenvalues and heterodimensional cycle: the proof of Proposition 4.3

Let us prove Proposition 4.3. The proofs of (1) and (2) have similar structures. We take a periodic point and modify the behavior of the diffeomorphism near the periodic point in the center-stable direction to obtain the desired property. The modification itself is simple but we need careful control the differentials in order to guarantee that the modification does not destroy the partially or volume hyperbolic structure. In this subsection, we prove Proposition 4.3 except the fact that the modification can be done preserving the volume hyperbolicity. Since the proof of it involves subtle linear algebraic argument, we will discuss it in the next subsection.

### 4.3.1 Auxiliary results

We prepare auxiliary lemmas which will be used in the proof. The following lemma guarantees the existence of convenient coordinates around a periodic point in a partially hyperbolic filtrating Markov

partition.

**Lemma 4.6.** *Let $\mathbf{R} = \bigcup R_i$ be a partially hyperbolic filtrating Markov partition (of saddle or attracting type) of $f$, $q \in \mathbf{R}$ a hyperbolic periodic point of s-index 2 and $\mathcal{C}^u$ a strictly invariant unstable cone field over $\mathbf{R}$. Then, there exists a $C^1$-coordinate neighborhood $(\varphi, U)$ with a coordinate system $(x, y, z)$ around $q$ such that the following holds:*

- *the local stable manifold of $q$ is equal to the $xy$-plane;*
- *the local unstable manifold of $q$ is equal to the $z$-axis;*
- *for every $\bar{x} \in U$, the cone field $(D\varphi)(\mathcal{C}^u_{\bar{x}})$ contains $z$-direction and is transverse to the $xy$-plane;*
- *for every $\bar{x} \in U$, the cone field $(D\varphi)\left(Df(\mathcal{C}^u_{f^{-1}(\bar{x})})\right)$ is well-defined and contains the $z$-direction (note that because of the invariance we also know that this cone field is transverse to the $xy$-plane).*

The proof of this lemma is easy. So we omit it.

**Remark 4.7.** *In Lemma 4.6, if $q$ has two different real contracting eigenvalues, we can choose the coordinate so that the local strong-stable manifold of $q$ coincides with the $x$-axis. Furthermore, by taking a linear coordinate change and restricting it to a small neighborhood of $q$, we can also assume that the weak stable direction at $q$ is equal to the $y$-direction.*

The following lemma guarantees the existence of a function having convenient behavior for the construction.

**Lemma 4.8.** *For every $K > 1$ and $\delta \in (0, 1)$, there exists $\alpha_0 \in (0, 1)$ such that for any $0 < \alpha < \alpha_0$ there is a smooth diffeomorphism $\chi = \chi_{K,\delta,\alpha} : [-1, 1] \to [-1, 1]$ satisfying the following:*

- $\chi(z) = Kz$ *for* $z \in [-\alpha, \alpha]$;
- $1 - \delta \leq \dot{\chi}(z) \leq K$ *for every* $z \in [-1, 1]$, *where* $\dot{\chi}(z)$ *is the first derivative of* $\chi(z)$;
- *for* $z$ *near* $\pm 1$, $\chi(z) = z$;
- $\max_{z \in [-1,1]} |\chi(z) - z| \leq \alpha K$.

*Proof.* Take a continuous piecewise linear function

$$P(z) = \begin{cases} Kz & (z \in [0, \alpha]), \\ ((1 - K\alpha)/(1 - \alpha))(z - 1) + 1 & (z \in [\alpha, 1]), \end{cases}$$

and define $P(z) := -P(-z)$ for $z \in [-1, 0]$. If $\alpha$ is sufficiently small, we can see that $P$ satisfies all the conditions we claimed except at $z = \pm\alpha$ and near $z = \pm 1$. Then, by removing the corner at $z = \pm\alpha$ and flatten it around $z = \pm 1$ appropriately, we obtain the function. $\square$

### 4.3.2 Local modifications (I): preservation of cones

Proposition 4.3 announces the existence of modifications in the center-stable direction keeping the partially hyperbolic and volume hyperbolic structures. In this subsection we give a general modification result preserving these structures. The modifications will be done in local coordinates: in this section we work on $\mathbb{R}^3$.

First, we fix a bump function

$$\rho(t) : \mathbb{R} \to [0, 1], \quad \text{satisfying} \quad \rho|_{[-1/2,1/2]} = 1 \text{ and } \rho(t) = 0 \text{ for } |t| > 1.$$

We fix a constant $C_\rho$ which is an upper bound of the derivative of $\rho$.

We start from the following definition.

**Definition 4.9.** *We say that a one-parameter family $\{\Gamma_t\}_{t \in [0,1]}$ of diffeomorphisms of $\mathbb{R}^2$ is a* modification family *if it satisfies the following properties:*

- *every $\Gamma_t$ is supported in the unit disc $\mathbb{D}^2$;*
- *the family $\{\Gamma_t\}_{t\in[0,1]}$ is jointly smooth with respect to $x, y$ and $t$;*
- $\Gamma_0 = \mathrm{Id}_{\mathbb{R}^2}$*;*
- *for every $t \in [0,1]$, $\Gamma_t$ preserves the origin, that is, $\Gamma_t(0,0) = (0,0)$ for every $t$.*

*We say that the modification family $\{\Gamma_t\}$ is* area preserving *if for every $t \in [0,1]$ the diffeomorphism $\Gamma_t$ is area preserving.*

Given a modification family $\{\Gamma_t\}_{t\in[0,1]}$, by putting $\tilde{\Gamma}(x,y,z) := (\Gamma_{\rho(z)}(x,y), z)$, we obtain a diffeomorphism of $\mathbb{R}^3$, supported on $\mathbb{D}^2 \times [-1,1]$ and $\tilde{\Gamma}|_{\mathbb{D}^2\times\{0\}} = \Gamma_1$. However, it may be that $\tilde{\Gamma}$ has large differentials and that may cause the destruction of hyperbolic structures. The following lemma shows that we can realize the behavior of $\Gamma_1$ in the center-stable direction keeping hyperbolic structures, by adding an extra modification.

**Lemma 4.10.** *Let $\mathcal{C}^u_1, \mathcal{C}^u_2$ be two cones of $\mathbb{R}^3$ strictly containing the $z$-direction, transverse to the $xy$-plane, and $\mathcal{C}^u_1$ is strictly contained in $\mathcal{C}^u_2$. Let $\{\Gamma_t\}_{t\in[0,1]}$ be a modification family. Then, given $\eta > 0$ there exists a diffeomorphism $\hat{\Gamma} : \mathbb{R}^3 \to \mathbb{R}^3$ such that the following holds:*

*(1) $\hat{\Gamma}$ is supported on $\mathbb{D}^2 \times [-1,1]$;*

*(2) for every $\bar{x} \in \mathbb{R}^3$, $D\hat{\Gamma}(\bar{x})(\mathcal{C}^u_1)$ is strictly contained in $\mathcal{C}^u_2$;*

*(3) for every $\bar{x} \in \mathbb{R}^3$ and every unit vector $u \in \mathcal{C}^u_2(\bar{x})$, we have $\|D\hat{\Gamma}(\bar{x})(u)\| > 1-\eta$ (note that the same holds for every unit vector in $\mathcal{C}^u_1$);*

*(4) $\hat{\Gamma}$ preserves the $xy$-plane (that is, the plane $\{z=0\}$). Furthermore, the restriction $\hat{\Gamma}|_{\mathbb{R}^2\times\{0\}}$ is conjugated to $\Gamma_1$ by a homothety of $\mathbb{R}^2$;*

*(5) $\hat{\Gamma}$ preserves the $z$-axis (note that this, together with the third item, implies that the restriction of $\hat{\Gamma}$ to the $z$-axis has its derivative greater than $1-\eta$).*

*Furthermore, if the modification family $\{\Gamma_t\}$ is area preserving, then we can choose $\hat{\Gamma}$ such that the following holds:*

$$|J_{xy}\hat{\Gamma} - 1| < \eta,$$

*where $J_{xy}\hat{\Gamma}$ is the determinant of the differential restricted to the $xy$-plane.*

*Proof.* First, remember that $\tilde{\Gamma}(x,y,z) := \big(\Gamma_{\rho(z)}(x,y), z\big)$ is a smooth diffeomorphism of $\mathbb{R}^3$ supported on the cylinder $\{(x,y,z) \mid r \le 1, z \in [-1,1]\}$, where $r := \sqrt{x^2+y^2}$. The diffeomorphism $\tilde{\Gamma}$ preserves the $z$ coordinate, hence its derivative at each point preserves the $xy$-plane.

We consider now the diffeomorphism of $\mathbb{R}^3$ obtained by conjugating it by a homothety:

$$\tilde{\Gamma}_\varepsilon = H_\varepsilon \circ \tilde{\Gamma} \circ (H_\varepsilon)^{-1},$$

where $H_\varepsilon(x,y,z) := (\varepsilon x, \varepsilon y, \varepsilon z)$ and $\varepsilon$ is a constant in $(0,1)$. By definition, $\tilde{\Gamma}_\varepsilon$ is a diffeomorphism of $\mathbb{R}^3$ supported on the cylinder $\{r \le \varepsilon, |z| \le \varepsilon\}$. It preserves the $z$ coordinate and its derivative at a point $(x,y,z)$ is that of $\tilde{\Gamma}$ at $\frac{1}{\varepsilon}(x,y,z)$. In particular, the derivative preserves the $xy$-planes and its norm is uniformly bounded independently of the choice of $\varepsilon$.

Now, for any $K > 1$, $\delta \in (0,1)$, and small $\alpha > 0$, we define a map

$$L_{K,\delta,\alpha}(x,y,z) := (x, y, \rho(r)\chi_{K,\delta,\alpha}(z) + (1-\rho(r))z),$$

where $\chi_{K,\delta,\alpha}$ is a map which satisfies all the properties in Lemma 4.8.

For any real number $K$, we denote the linear map $(x,y,z) \mapsto (x,y,Kz)$ by $E_K$. Note that the following holds:

- $L_{K,\delta,\alpha}$ coincides with $E_K$ on the cylinder $\left\{r \le \frac{1}{2}, z \in [-\alpha,\alpha]\right\}$;
- the derivative $DL_{K,\delta,\alpha}(\bar{x})$ preserves the $z$-direction for every $\bar{x} \in \mathbb{R}^3$;

- the derivative $DL_{K,\delta,\alpha}(\bar{x})$ preserves the $xy$-plane for $\bar{x}$ in the (infinite) cylinder $\left\{r \leq \frac{1}{2}\right\}$.

Then, given a modification family $\{\Gamma_t\}$, we define a diffeomorphism $\hat{\Gamma}_{K,\delta,\alpha}$ as follows:

$$\hat{\Gamma}_{K,\delta,\alpha}(x,y,z) := L_{K,\delta,\alpha} \circ \tilde{\Gamma}_{\alpha/2}.$$

The map $\hat{\Gamma}_{K,\delta,\alpha}$ is a smooth diffeomorphism of $\mathbb{R}^3$ supported on the cylinder $\{r \leq 1, z \in [-1,1]\}$. By construction, we can see that $\hat{\Gamma}_{K,\delta,\alpha}$ satisfies the conditions (1), (4) and (5) in Lemma 4.10. Now we choose the parameters $K, \delta$ and $\alpha$ so that $\hat{\Gamma}_{K,\delta,\alpha}$ satisfies the rest of the properties. To see this, we calculate the derivative of $\hat{\Gamma}_{K,\delta,\alpha}$.

First, we calculate $D\tilde{\Gamma}_{\alpha/2} = D(H_{\alpha/2} \circ \tilde{\Gamma} \circ (H_{\alpha/2})^{-1})$. Since the conjugation by a homothety does not have any effect on the calculation of the derivative, it can be written as follows:

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{A(\frac{2}{\alpha}\bar{x})} & B(\frac{2}{\alpha}\bar{x}) \\ \hline 0 & 0 & 1 \end{array}\right),$$

where $A := D_{xy}\Gamma_{\rho(z)}$, $B := \dot{\rho}(z)(D_z\Gamma_{\rho(z)})$. Note that these two matrices have bounded norms for fixed choice of the family $\{\Gamma_t\}$ and $\rho$. By this calculation we can see that this differential leaves the $xy$-plane invariant. Furthermore, we can also see that if $\Gamma_t$ is area preserving, then this differential preserves the area of the $xy$-plane.

We calculate the derivative $DL_{K,\delta,\alpha}(\bar{x})$. It can be written as follows:

$$\left(\begin{array}{c|c} \mathrm{Id} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline C(\bar{x}) & d(\bar{x}) \end{array}\right)$$

where Id denotes the two dimensional identity matrix,

$$C(\bar{x}) = \left(\partial_x(\rho(r))(\chi(z) - z),\, \partial_y(\rho(r))(\chi(z) - z)\right),$$

and $d(\bar{x}) = \rho(r)\dot{\chi}(z) + (1 - \rho(r))$. By definition of $\chi$ and $C_\rho$, one can see that the following inequalities hold:

$$\|C(\bar{x})\| \leq 2\alpha K C_\rho, \quad 1 - \delta \leq d(\bar{x}) \leq K.$$

Now, we calculate $D\hat{\Gamma}_{K,\delta,\alpha}$.

- If $\bar{x} \in \{r \leq \frac{\alpha}{2}, |z| \leq \frac{\alpha}{2}\}$, then $\bar{x}$ is in the support of $\tilde{\Gamma}_{\alpha/2}$ and $DL_{K,\delta,\alpha} = E_K$. Hence, the derivative $D\hat{\Gamma}_{K,\delta,\alpha}$ is
$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{A(\frac{2}{\alpha}\bar{x})} & B(\frac{2}{\alpha}\bar{x}) \\ \hline 0 & 0 & K \end{array}\right). \tag{1}$$

- If $\bar{x} \in \{r \leq \frac{\alpha}{2}, |z| \geq \frac{\alpha}{2}\}$, then $\bar{x}$ is outside the support of $\tilde{\Gamma}_{\alpha/2}$. Furthermore, we know that $\rho \equiv 1$ on this region. Thus the derivative $D\hat{\Gamma}_{K,\delta,\alpha}$ is
$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{Id}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & \dot{\chi}(z) \end{array}\right). \tag{2}$$
Recall that by definition we have $1 - \delta \leq \dot{\chi}(z) \leq K$.

- If $\bar{x} \in \{r \geq \frac{\alpha}{2}\}$, $\hat{\Gamma}_{K,\delta,\alpha} = L_{K,\delta,\alpha}$. Hence the derivative $D\hat{\Gamma}_{K,\delta,\alpha}$ is
$$\left(\begin{array}{c|c} \mathrm{Id} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline C(\bar{x}) & d(\bar{x}) \end{array}\right). \tag{3}$$
Recall that we have $\|C(\bar{x})\| \leq 2\alpha K C_\rho$ and $1 - \delta \leq d(\bar{x}) \leq K$.

By these calculations, we can conclude the following:

**Claim 2.** *Let $\mathcal{C}_1^u, \mathcal{C}_2^u$ be two cones of $\mathbb{R}^3$ strictly containing the $z$ direction, transverse to the $xy$-plane. Assume that $\mathcal{C}_1^u$ is strictly contained in $\mathcal{C}_2^u$.*

*Then, for any $\eta > 0$, there is $K > 1$ and $\delta_0 \in (0,1)$ such that the following holds: for any $0 < \delta < \delta_0$, there is $0 < \alpha_0 \le \alpha(K,\delta)$ such that for any $0 < \alpha < \alpha_0$ the diffeomorphism $\hat{\Gamma}_{K,\delta,\alpha}$ satisfies the conditions (2) and (3) in the conclusion of Lemma 4.10. Namely:*

- *For any $\bar{x} \in \mathbb{R}^3$, the image $D\hat{\Gamma}_{K,\delta,\alpha}(\bar{x})(\mathcal{C}_1^u)$ is strictly contained in $\mathcal{C}_2^u$;*
- *For any $\bar{x} \in \mathbb{R}^3$ and any unit vector $u \in \mathcal{C}_2^u$ the norm of $D\hat{\Gamma}_{K,\delta,\alpha}(\bar{x})(u)$ is larger than $1-\eta$:*

$$\|D\hat{\Gamma}_{K,\delta,\alpha}(\bar{x})(u)\| > 1 - \eta.$$

*Proof of Claim 2.* Let us check that the announced properties hold on each domain.

- If $\bar{x}$ belongs to the support of $\tilde{\Gamma}_{\alpha/2}$, that is, the region $\{r \le \frac{\alpha}{2}, |z| \le \frac{\alpha}{2}\}$, then the derivative $D\hat{\Gamma}_{K,\delta,\alpha}(\bar{x})$ is given by (1). As $A(\frac{2}{\alpha}\bar{x})$ and $B(\frac{2}{\alpha}\bar{x})$ are uniformly bounded (independently of the choice of $\alpha$), we see that $D\hat{\Gamma}_{K,\delta,\alpha}(\bar{x})(\mathcal{C}_1^u)$ is strictly contained in $\mathcal{C}_2^u$ for $K$ large enough. Indeed, as $K$ goes to $+\infty$, the image $D\hat{\Gamma}_{K,\delta,\alpha}(\bar{x})(\mathcal{C}_1^u)$ tends to the $z$ direction. Furthermore, for $K$ large, any vector in $\mathcal{C}_2^u$ is expanded, in particular, each unit vector has its image longer than $1-\delta$.
- If $\bar{x} \in \{r \le \frac{\alpha}{2}, |z| \ge \frac{\alpha}{2}\}$, then the derivative $D\hat{\Gamma}_{K,\delta,\alpha}(\bar{x})$ is given by (2). It is a diagonal matrix having the identity matrix in the $xy$-direction, and in the $z$ direction it is a multiplication at least by $1-\delta$. Thus by choosing $\delta$ small enough, we have that $D\hat{\Gamma}_{K,\delta,\alpha}(\bar{x})(\mathcal{C}_1^u)$ is contained in a cone arbitrarily close to $\mathcal{C}_1^u$, hence contained in $\mathcal{C}_2^u$. Furthermore, the vectors in $\mathcal{C}_2^u$ are expanded at least by $1-\delta$. Hence, by choosing $\delta$ to be smaller than $\eta$, one gets the second condition.
- If $\bar{x} \in \{r \ge \frac{\alpha}{2}\}$, then the derivative is given by (3). For fixed $K$ and $\delta$, when $\alpha$ goes to 0 the non-diagonal term $C(\bar{x})$ tends uniformly to 0 (indeed, it is bounded by $2\alpha K C_\rho$). Thus for $\alpha$ close to 0, the derivative of $\hat{\Gamma}$ almost preserves the $xy$-plane. The diagonal terms are the identity on the $xy$-plane and $d(\bar{x})$ in the $z$ direction which satisfies $1-\delta \le d(\bar{x}) \le K$. Thus we can get the conclusion as in the second item.

As a result, given a constant $\eta$, by choosing $K$, $\delta$ and $\alpha$ appropriately in the order explained above, we have the desired properties. □

Finally, if $\{\Gamma_t\}$ is area preserving, then by choosing $\alpha$ sufficiently small, we can assume that the determinant of $\hat{\Gamma}$ restricted to the $xy$-plane is arbitrarily close to 1. To be more precise, the term $C(\bar{x})$ is the unique trouble for the determinant not being exactly 1, but we have seen that $C(\bar{x})$ goes to 0 when $\alpha$ tends to 0. Thus the proof of Lemma 4.10 is completed. □

### 4.3.3 Local modifications (II): existence of convenient families

In this subsection, we discuss the existence of modification families with convenient properties for the construction.

**Lemma 4.11.** *There exists a modification family $\{\Gamma_t\}$ of $\mathbb{R}^2$ which is area preserving and satisfies one of the following conditions:*

*(1) $\|\Gamma_t(x,y)\| = \|(x,y)\|$ for every $t \in [0,1]$ and every $(x,y) \in \mathbb{R}^2$, where $\|\cdot\|$ denotes the distance from the origin. Furthermore, $D_{xy}\Gamma_1(x,y)$ is a rotation matrix of angle $\pi/2$ for every $(x,y)$.*

*(2) $\Gamma_1(x,y)$ preserves $x$-axis and $\|\Gamma_1(x,0)\| \le \|(x,0)\|$ for every $x$. Furthermore, $D_{xy}\Gamma_1(0,0) = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$, where $\beta > 1$ is a constant (which we can choose).*

*Proof.* For the construction of modification family $\{\Gamma_t\}$ satisfying the condition (1) and area preserving property, we take a Hamiltonian function $\Xi(x,y) = (x^2 - y^2)\rho(r)$, where $\rho$ is a smooth bump function defined in the previous section. By $\{\Omega_t\}$ we denote the time $t$ map of corresponding Hamiltonian vector field. Now $\Gamma_t := \Omega_{(\pi/2)t}$ gives us the desired family.

For (2), consider the Hamiltonian function $\Xi(x,y) = xy \cdot \rho(r)$ and put $\Gamma_t := \Omega_{\log(\beta)t}$. □

### 4.3.4 Realizing the diffeomorphisms of $\mathbb{R}^3$ as the modification of $f$

Let us start the proof of Proposition 4.3.

*Proof of (1).* Let $\mathbf{R} = \bigcup R_i$ be a partially hyperbolic filtrating Markov partition of $f$ satisfying all the hypotheses and let $\mathcal{C}^u$ be a strictly $Df$-invariant unstable cone field on $\mathbf{R}$. Since $H(p)$ is non-trivial and $\mathbf{R}$ is a filtrating set, there exists a hyperbolic periodic point $p_1 \in \mathbf{R}$ homoclinically related with $p$. We can assume that $Df^{\pi(p_1)}(p_1)|_{E^s}$ has positive determinant (see for example [BDP, Lemma 4.16]). If $Df^{\pi(p_1)}(p_1)|_{E^s}$ has a non-real eigenvalue, then we are done. So suppose that it does not. By applying Lemma 4.6, we choose local coordinates $\varphi$ in a neighborhood $U$ of $p_1$ in which the local invariant manifolds of $p_1$ are flat. By taking a linear coordinate change around $p_1$, we can assume that the two different stable eigendirections of $Df^{\pi(p_1)}(p_1)| = E^s$ are orthogonal.

Let $\mathcal{C}^u_2$ and $\mathcal{C}^u_1$ be the image of the unstable cone $\mathcal{C}^u(p_1)$ and of $Df(\mathcal{C}^u(f^{-1}(p_1)))$ under $D\varphi$, respectively. Take $\eta > 0$ sufficiently close to 1 such that $(1-\eta)^{-1}$ is strictly less than a lower bound $\mu$ for the rate of the expansion of vectors in the unstable cone $\mathcal{C}^u(p_1)$. We apply Lemma 4.10 to these cones, the family of diffeomorphisms $\{\Gamma_t\}$ in Lemma 4.11 (1) and $\eta$ to obtain the diffeomorphism $\hat{\Gamma}$.

Notice that the same conclusion of Lemma 4.10 holds for any pair of cones sufficiently close to $\mathcal{C}^u_1$ and $\mathcal{C}^u_2$. We choose $\varepsilon > 0$ such that for any $\bar{x} \in [-\varepsilon, \varepsilon]^3$ the cones $D\varphi(\mathcal{C}^u(\varphi^{-1}(\bar{x})))$ and $D\varphi(Df(\mathcal{C}^u(\varphi^{-1}(\bar{x}))))$ are close enough to $\mathcal{C}^u_2$ and $\mathcal{C}^u_1$ respectively so that the conclusions of Lemma 4.10 still holds.

Now we define $g_0$ to be the diffeomorphism which coincides with $\varphi^{-1} \circ (H_\varepsilon \circ \hat{\Gamma} \circ (H_\varepsilon)^{-1}) \circ \varphi \circ f$ on $f^{-1}(U)$ and with $f$ outside. By construction, the cone field $\mathcal{C}^u$ is strictly invariant under $Dg_0$ and $Dg_0$ expands the vectors by a factor at least $(1-\eta)\mu > 1$. So the cone field $\mathcal{C}^u$ is not only strictly invariant but also unstable. Notice that for sufficiently small $\varepsilon$ the orbit of $p_1$ meets the support of $\varphi^{-1} \circ (H_\varepsilon \circ \hat{\Gamma} \circ (H_\varepsilon)^{-1}) \circ \varphi$ only at the point $p_1$. Then a simple calculation shows that $p_1$ has stable complex eigenvalues.

Let us show that if $\varepsilon$ sufficiently small, then $p_1$ keeps the homoclinic relation with $p$. Indeed, in our coordinates, the local stable manifold $W^s_{\mathrm{loc}}(p_1, f)$ of $p_1$ is exactly equal to the $xy$-plane, which is preserved by this modification. For $\varepsilon$ small, the support of the modification does not intersect the images $f^i(W^s_{\mathrm{loc}}(p_1, f))$ for $0 < i < \pi(p_1)$. Thus $W^s_{\mathrm{loc}}(p_1, f) = W^s_{\mathrm{loc}}(p_1, g_0)$ for every $\varepsilon$. Similarly, we have $W^u_{\mathrm{loc}}(p_1, f) = W^u_{\mathrm{loc}}(p_1, g_0)$. Let $s^+ \in W^s_{\mathrm{loc}}(p_1, f) \cap W^u(p, f)$ and $s^- \in W^u_{\mathrm{loc}}(p_1, f) \cap W^s(p, f)$ be heteroclinic points. For $\varepsilon$ small enough the support of modification is disjoint from the negative orbit of $s^+$ and of the positive orbit of $s^-$. Thus $s^+$ and $s^-$ are still heteroclinic points between $p$ and $p_1$.

Finally, since the support of the perturbation is contained in an arbitrarily small neighborhood of $p_1$, this neighborhood can be chosen so that it is disjoint from $W^s_{\mathrm{loc}}(p, f)$, and hence $W^s_{\mathrm{loc}}(p, f) = W^s_{\mathrm{loc}}(p, g_0)$. Thus, we can see that $p$ still has large stable manifold. Thus the proof is completed. $\square$

Let us see the second modification.

*Proof of (2).* We start from $f$ satisfying the conclusion of item (1). Remember that $p_1$ is a periodic point with (non-real) complex stable eigenvalues and $p$ is a hyperbolic periodic point homoclinically related with $p_1$ having large stable manifold.

According to [BDP, Proposition 2.5], an arbitrarily $C^1$-small perturbation of $f$ produces a hyperbolic periodic point $q$ of $s$-index 2, homoclinically related with $p$ and $p_1$, and whose derivative in the period restricted to the center-stable space is a contracting homothety. We fix a point of transversal heteroclinic intersection $x \in W^s(q) \cap W^u(p)$. By an arbitrarily small perturbation of $f$ in an arbitrarily small neighborhood of $q$ we can change slightly the derivative of $q$ in the period in such a way that it has two real eigenvalues of different moduli and (the continuation of) $x$ still belongs to the strong stable manifold $W^{ss}(q)$ of $q$.

Thus we now assume that $f$ itself has these properties. More precisely, $f$ has a periodic point $q$ homoclinically related with $p$, with two stable real eigenvalues of different moduli, and a point $x$ of heteroclinic intersection belonging to the strong stable manifold of $q$.

Then by applying Lemma 4.6, together with Remark 4.7, we take a local coordinate $\phi\colon V \to \mathbb{R}^3$ around $q$ such that the local stable manifold coincides with the $xy$-plane, the local strong stable manifold coincides with the $x$-axis, the $y$ direction is the weak stable direction and the local unstable manifold coincides with the $z$-axis.

Now we perform the modification as in the proof of (1): we use Lemma 4.10 for the cones $D\phi(\mathcal{C}^u(q))$ and $D\phi\left(Df(\mathcal{C}^u(f^{-1}(q)))\right)$, $\eta$ as in (1) and $\{\Gamma_t\}$ as in Lemma 4.11(2) letting $\beta$ greater than the weak stable eigenvalue of $q$ to obtain the diffeomorphism $\hat{\Gamma}$ of $\mathbb{R}^3$. Using this $\hat{\Gamma}$, we modify $f$ as follows: $\phi^{-1} \circ (H_\varepsilon \circ \hat{\Gamma} \circ (H_\varepsilon)^{-1}) \circ \phi \circ f$ on $f^{-1}(V)$ and keep intact outside.

As $\beta$ is bigger than the inverse of the weak stable eigenvalue of $q$, we see that $q$ is now an $s$-index 1 hyperbolic saddle. By the similar argument as is in the proof of (1), by choosing sufficiently small $\varepsilon$, we can check the preservation of partially hyperbolic filtrating Markov partition structure, the largeness of the stable manifold through $p$ and the preservation of heteroclinic relation between $p$ and $q$. Once again by decreasing $\varepsilon$ if necessary, we can avoid the interference of the modification to the (transverse) homoclinic intersection between $p$ and $p_1$. Thus for sufficiently small $\varepsilon > 0$ the modification above gives us the desired diffeomorphism $g$. $\square$

**Remark 4.12.** *While the modification above needs to be $C^1$-large in general, by shrinking $\varepsilon$, we can assume that this modification is arbitrarily $C^0$-small.*

## 4.4 Preservation of volume hyperbolicity

In this section we will prove the last part of Proposition 4.3. That is, we show that if the initial diffeomorphism $f$ is volume hyperbolic then we can perform the modification keeping the volume hyperbolicity. This will follow from the following two properties:

- the modification we perform can be done so that the unstable bundle is almost preserved (this can be guaranteed by replacing the unstable cone field $\mathcal{C}^u$ by high forward image $Df^n(\mathcal{C}^u)$).
- the modifications we perform almost preserve the area of the horizontal subspaces.

### 4.4.1 Normal bundles and volume hyperbolicity

Let us explain why these two properties guarantee the preservation of volume hyperbolicity. Since we already proved the invariance of the cone field and the uniform expansion of the vectors in the unstable cone in the previous subsection, we certainly have the partial hyperbolicity, in particular, uniform expansion property of the unstable bundle. It remains to prove the uniform contraction of the area in the center-stable bundle. This is not so easy because we do not know the position of the center-stable bundle after the modification: the only fact available is that it is transverse to the unstable cone. we need to see how we can control the center-stable determinant without controlling precisely the center-stable plane.

To see this, we consider the following general situation. Let $f$ be a diffeomorphism of $M$ and $K \subset M$ be a compact $f$-invariant set. We assume that $K$ is a maximal invariant set of some neighborhood $U$ of $K$ and it is partially hyperbolic with a uniformly expanding, one-dimensional unstable bundle and a two-dimensional center-stable bundle $T_K M = E^{cs} \oplus E^u$. Furthermore, we assume that $K$ is also volume hyperbolic, that is, the determinant of $Df$ restricted to the center-stable bundle is uniformly contracting. Let $\mathcal{C}^u$ be a strictly invariant unstable cone field defined on $U$, strictly containing the unstable direction and transverse to the center-stable direction at every point of $K$. By definition we also know that the vectors in $\mathcal{C}^u$ are uniformly expanding. Such a cone field always exists, by shrinking the neighborhood $U$ if necessary (see Lemma 2.3).

Let $\mathcal{P} = \{P(x) \mid x \in U, P_x \subset T_x M\}$ be a continuous distribution of the same dimension as that of center-stable bundle of $K$, and is transverse to $\mathcal{C}^u(x)$. Note that $\mathcal{P}$ defines a vector bundle over $U$.

Now for $x \in K$ we define a linear map $D(P, f)(x) \colon P(x) \to P(f(x))$ to be the one obtained as the composition of $Df(x)|_{P(x)} \colon P(x) \to T_{f(x)}M$ and $\mathrm{Proj}_{\|E^u(f(x))} P(f(x))$ (for two complementary vector subspaces $V$ and $W$ in a vector space, by $\mathrm{Proj}_{\|V} W$ we denote the projection from $V \oplus W$ to $W$ along to $V$). The collection of maps $D(P, f)(x)$ defines a linear cocycle $\mathcal{D}_{\mathcal{P},f}$ on the bundle obtained restricting $\mathcal{P}$ over $K$.

**Remark 4.13.** *The linear cocycle $\mathcal{D}_{\mathcal{P},f}$ is conjugated to the restriction of $Df$ to the center-stable bundle $E^{cs}$ by a continuous bundle map inducing the identity map on the base space $K$ (i.e., a continuous family of linear maps $E^{cs}(x) \to P(x)$, $x \in K$). Indeed, this map is given by the projection along $E^u(x)$.*

*By the existence of the conjugation, we see that $\mathcal{D}_{\mathcal{P},f}$ is uniformly volume contracting. Thus we can choose a metric on $M$ and a constant $\lambda$ satisfying $0 < \lambda < 1$ such that for any $x \in K$ the determinant of the linear map $D(P, f)(x)$ with respect to orthonormal basis has absolute value bounded by $\lambda$ from above.*

#### 4.4.2 Modification and normal bundle

Now, let us consider the effect of modifications in this setting. Let $g = h \circ f$ be a diffeomorphism such that $g$ strictly leaves the cone field $\mathcal{C}^u$ invariant on $U$, expands uniformly the vectors in $\mathcal{C}^u$. We assume that the maximal invariant set $K_g$ of $g$ in $U$ is contained in a small neighborhood of $K$. We also assume that the unstable bundle $E^u_g$ is very close to $E^u = E^u_f$. More precisely, we require that for every $x \in K_g$ there exists a point $y \in K$ close to $x$ so that $E^u_g(x)$ is close $E^u(x)$.

Let us consider the situation where $h$ almost preserves the bundle $\mathcal{P}$ and that the determinant of the restriction $Dh|_{\mathcal{P}}$ is very close to 1. In this setting, we have the following:

**Claim 3.** *Under the above hypotheses, $g$ is volume hyperbolic on $K_g$.*

*Proof.* We take a linear cocycle $\mathcal{D}_{\mathcal{P},g} = \{D(P, g)(x) \mid x \in K_g\}$, where

$$D(P, g)(x) := \left(\mathrm{Proj}_{\|_{Df(E^u_g(x))}} P(g(x))\right) \circ Dg(x)|_{P(x)}.$$

Fix some constant $\lambda_1$ satisfying $\lambda < \lambda_1 < 1$. We show that the determinant of $D(P, g)(x)$ has its absolute value bounded by $\lambda_1$ for every $x \in K_g$.

Given $x \in K_g$, by assumption, there exists a point $y \in K$ close to $x$ such that $E^u_g(x)$ is close to $E^u_f(y)$. Thus $Df(E^u_g(x))$ is close to $DfE^u_f(y) = E^u_f(f(y))$. Furthermore, since $x$ is close to $y$, we see that $P(f(x))$ is close to $P(f(y))$. Combining these, we see that the map $\left(\mathrm{Proj}_{\|_{Df(E^u_g(x))}} P(f(x))\right) \circ Df|_{P(x)}$ is very close to $D(P, f)$ and therefore its determinant is almost bounded by $\lambda$.

Now recall that $Dh$ almost preserves $\mathcal{P}$. This implies that $Dh(P(f(x)))$ is very close to $P((h \circ f)(x)) = P(g(x))$. As a consequence, the determinant the restriction of $\left(\mathrm{Proj}_{\|_{E^u_g(g(x))}} P(g(x))\right)$ to $Dh(P(f(x)))$ is very close to 1. Furthermore, the determinant of the restriction of $Dh$ to $P(f(x))$ is assumed to be almost 1.

As a result, we deduce that the determinant of

$$\left(\mathrm{Proj}_{\|_{E^u_g(g(x))}} P(g(x))\right) \circ Dh \circ \left(\mathrm{Proj}_{\|_{Df(E^u_g(x))}} P(f(x))\right) \circ Df|_{P(x)}$$

is bounded by some constant $\lambda_1 < 1$ (which can be chosen independently of $x$).

Then, notice that we have the following equality:

**Claim 4.** *For every $x \in K_g$, we have*

$$D(P, g)(x) = \left(\mathrm{Proj}_{\|_{E^u_g(g(x))}} P(g(x))\right) \circ Dh \circ \left(\mathrm{Proj}_{\|_{Df(E^u_g(x))}} P(f(x))\right) \circ Df|_{P(x)}.$$

*Proof.* This is a consequence of the following the general fact: let $U_i = V_i \oplus W_i$, $i = 1, 2, 3$ be vector spaces and $F : U_1 \to U_2$, $H : U_2 \to U_3$ and $G : U_1 \to U_3$ are linear maps satisfying $G = H \circ F$. Assume that $F(W_1) = W_2$ and $H(W_2) = W_3$. Then

$$\left(\mathrm{Proj}_{\|_{W_3}} V_3\right) \circ G = \left(\mathrm{Proj}_{\|_{W_3}} V_3\right) \circ H \circ \left(\mathrm{Proj}_{\|_{W_2}} V_2\right) \circ F.$$

The equality that we want to show is the direct consequence of this general result. □

Since the determinant of the right hand side of Claim 4 is bounded by $\lambda_1$, so is the left hand side. □

### 4.4.3 Conclusion of volume hyperbolicity

Now, let us finish the proof of Proposition 4.3.

*Proof of the last part of Proposition 4.3.* We perform the modification $g = h \circ f$ of Proposition 4.3 to higher forward refinement of $\mathbf{R} = \bigcup R_i$. Since, by taking forward iteration, the unstable cone $\mathcal{C}^u$ converges to the unstable direction at each point of the maximal invariant set of $\mathbf{R}$, we can assume that the modification in Proposition 4.3 preserves the unstable direction as much as we want. Note that, by the last item of Lemma 4.10, we can assume that $h$ almost preserves the area of center-stable planes.

Now we take a two dimensional distribution $\mathcal{P}$ transverse to $\mathcal{C}^u$ which coincides with the $xy$-plane in the support of the modification. Remember that the modification $g = h \circ f$ can be done so that it is arbitrarily $C^0$-close to $f$, see Remark 4.12. As a result, we can assume that the maximal invariant set $K_g$ is contained in arbitrarily small neighborhood of that of $f$. Thus, Claim 3 ensures that for $g = h \circ f$ chosen as above has maximal invariant set which is volume hyperbolic. $\square$

## 4.5 On the proof of Corollary 1.4

Let us see how we prove Corollary 1.4.

In Proposition 4.1, Remark 4.2 and Proposition 4.3, we have seen that we can construct a local map $f$ on the three-dimensional ball $B^3$ satisfying the following:

- it is a diffeomorphism on its image;
- the ball is attracting, that is, $f(B^3) \subset \mathrm{int}(B^3)$ holds;
- $f$ contains a mixing partially hyperbolic Markov partition $\mathbf{R}$ of attracting type which satisfies the hypothesis of Theorem 1 having volume hyperbolicity on the maximal invariant set of $\mathbf{R}$;
- the whole ball $B^3$ is contained in the basin of attraction of $\mathbf{R}$, that is, for every $x \in B^3$ there exists $n > 0$ such that $f^n(x) \in \mathbf{R}$ holds.

Note that, because of these properties, the chain recurrence class $C(p)$ (remember that $p$ is the hyperbolic periodic point with large stable manifold) is the unique quasi-attractor in $B^3$ in $C^1$-robust way (see Corollary 2.27).

Now, for any 3-manifold $M$ we can construct a diffeomorphism without (topological) attractors or repellers as follows (see also section 4.2 of [BLY]): take a Morse function of $M$ and its gradient flow. Then replace each sink by $(B^3, f)$ and each source by $(B^3, f^{-1})$ and glue them appropriately. Now by Corollary 1.2 we have the conclusion: there are finitely many quasi-attractors and quasi-repellers and, for $C^1$-generic diffeomorphisms in the neighborhood, every quasi-attractor and quasi-repeller is wild.

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